

Existence of the thermodynamic limit and asymptotic behavior of some irreversible quantum dynamical systems

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B.S. (Kharkov National University, Kharkov, Ukraine) 2007

M.A. (University of California, Davis) 2012

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

in the

OFFICE OF GRADUATE STUDIES

of the

UNIVERSITY OF CALIFORNIA

DAVIS

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2012

To my mother, my constant supporter.

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Abstract

This dissertation discusses the properties of two open quantum systems with a general class of irreversible quantum dynamics. First we study Lieb-Robinson bounds in a quantum lattice systems. This bound gives an estimate for the speed of growth of the support of an evolved local observable up to an exponentially small error. In a second model we study the properties of a leaking cavity pumped by a random atomic beam.

We begin by describing quantum systems on an infinite lattice with associated finite or infinite dimensional Hilbert space. The generator of the dynamics of this system is of the Lindblad-Kossakowski type and consists of two parts: the Hamiltonian interactions and the dissipative terms. We allow both of them to be time-dependent. This generator satisfies some suitable decay condition in space. We show that the dynamics with a such generator on a finite system is a well-defined quantum dynamics in a sense of a norm-continuous cocycle of unit preserving completely positive maps.

Lieb-Robinson bounds for irreversible dynamics were first considered in [10] in the classical context and in [20] for a class of quantum lattice systems with finite-range interactions. We extend those results by proving a Lieb-Robinson bound for lattice models with a more general class of quantum dynamics.

Then we use Lieb-Robinson bounds for a finite lattice systems to prove the existence of the thermodynamic limit of the dynamics. We show that in a strong limit there exists a strongly continuous cocycle of unit preserving completely positive maps.

Which means that the dynamics exists in an infinite system, where Lieb-Robinson bounds also holds.

In the second part of the dissertation we consider a system that consists of a beam of two-level atoms that pass one by one through the microwave cavity. The atoms are randomly excited and there is exactly one atom present in the cavity at any given moment. We consider both the ideal and leaky cavity and study the time asymptotic behavior of the state of the cavity.

We show that the number of photons increases indefinitely in the case of the ideal cavity. In case of leaky cavity, we prove that the mean photon number in the cavity stabilizes in time. The limiting state of the cavity in this case exists and it is independent of the initial state. We calculate the characteristic functional of this non-quasi-free non-equilibrium state on the Weyl algebra. We also calculate the energy flux in both the ideal and open cavity and the entropy production for the ideal cavity.

Acknowledgments and Thanks

It is my great pleasure to thank my adviser Prof. Bruno Nachtergaele, who encouraged and challenged me throughout my years at the UC Davis. His patience and exceptional guidance helped me in my research and in completion of this dissertation.

I am very grateful to Prof. Valentin Zagrebnov, with whose invaluable collaboration we completed projects presented in this dissertation.

I would like to thank all faculty and staff of the Math department, who created a friendly working environment on the department.

Also I would like to thank my mother and my sister who always believed in me and gave me the needed support during the toughest times.

The work presented in this dissertation was supported by the National Science Foundation under grants DMS-075758, DMS-1009502, DMS-10009502, VIGRE DMS-0636297 and by the France-Berkeley Fund under project # 201013308.

Chapter 1

Introduction

1.1 Introduction

The study of open quantum systems is significant to various areas of application of quantum theory such as quantum optics, quantum information theory, atomic physics and condensed-matter physics. In this dissertation we assume an open system dynamics of a quantum system. First we study Lieb-Robinson bounds for a lattice system with irreversible time-dependent dynamics. Then we study the properties of a state of the microwave cavity pumped by a random atomic beam. The dynamics of the system is also time-dependent and irreversible.

In the past years Lieb-Robinson bounds have been shown to be a powerful tool to turn the locality properties of physical systems into useful mathematical estimates. Lieb-Robinson bounds provide an estimate for the speed of propagation of signals in a spatially extended system.

Lieb-Robinson bounds were so far considered on a lattice system (for example \mathbb{Z}^d). Briefly, if A and B are two observables supported in disjoint regions X and Y , respectively, (see section 2.2.2 for the definition of the support) on a lattice system

and if τ_t denotes the unitary time evolution of the system, a Lieb-Robinson bound is an estimate of the form

$$\|[\tau_t(A), B]\| \leq C e^{-\mu(d(X,Y)-v|t|)}, \quad (1.1.1)$$

where C, μ and v are positive constants and $d(X, Y)$ denotes the distance between X and Y . Here v is called Lieb-Robinson velocity. See Chapter 2.2.7 for more details.

From this inequality we see that for the small times $|t| < d(X, Y)/v$ the norm on the right-hand side is exponentially small. The reason we consider the commutator on the left-hand side of the Lieb-Robinson bounds is the following. The commutator between the observables A and B is zero if their supports are disjoint. The converse is also true: if the observable A is such that its commutator with any observable B supported outside some set X is zero, then A has a support inside the set X . This statement is also approximately true in the sense: suppose that there exist some $\epsilon > 0$ such that $\|[A, B]\| \leq \epsilon \|B\|$ for any observable B that is supported outside a set X and some observable A . Then there exists an observable $A^{(\epsilon)}$ with the support inside the set X that approximate the observable A : $\|A - A^{(\epsilon)}\| \leq \epsilon$.

So Lieb-Robinson bounds say that the time evolution of the observable A with the support in a set X is mainly supported in the δ -neighborhood of X , where $\delta > v|t|$ with v is the Lieb-Robinson velocity.

In quantum information if Alice has access to the observable A with support X and sends a signal to Bob who has access to the observable B with support Y then the strong enough signal that Bob can detect propagates with the Lieb-Robinson velocity v .

In 1972 Lieb and Robinson first showed the existence of the light-cone such that the amount of information signaled beyond it decays exponentially [12]. This result,

known as Lieb-Robinson bounds, has been improved in [9], [10], [14] -[18] , for example, making the bound on the speed independent of the dimension of the spin spaces and extending the class of Hamiltonians describing the system. In these works the Lieb-Robinson bounds was considered for the unitary evolution, when the dynamics is described by the Heisenberg equation.

We consider a more general situation, when the dynamics is described by a Markovian dynamical semigroup (see Chapter 2.2.4 for more details) with a time dependent generators. Lieb-Robinson bounds for irreversible dynamics were, to our knowledge, first considered in [10] in the classical context and in [20] for a class of quantum lattice systems with finite-range interactions. We extended these results by proving the Lieb-Robinson bound for a long-range exponentially decaying time dependent interactions.

One of the main applications of Lieb-Robinson bounds is the existence of the dynamics in the thermodynamic limit. The existence of the thermodynamics limit is important as a fundamental property of any model meant to describe properties of bulk matter. In particular, such properties should be essentially independent of the size of the system which, of course, in any experimental setup will be finite. Our results are described in Section 3.

In the second part of the thesis we use the results about the irreversible dynamics to study the properties of a leaking cavity pumped by a random atomic beam.

The micromaser system that we consider consists of a beam of two-level atoms that pass one by one through the microwave cavity. The atoms are excited with some probability p . The cavity is modeled by a single photon mode. The beam is tuned in such a way that there is exactly one atom in the cavity at any given time. While in the cavity the corresponding single atom is able to interact with a single mode of

the cavity field.

The interaction between the atom and the cavity in our model leaves the state of the atom invariant which may be interpreted as the limit of heavy atoms and soft photons. Hence, we focus our study on the time asymptotic behavior of the cavity state.

We do this both in the case of a perfect (non-leaky) cavity as well as in the case of a leaking cavity modeled by adding a Lindblad-Kossakowski dissipative term to the generator of the Hamiltonian dynamics (see section 2.2.4 for details). We show that in the case of the perfect cavity, the expected number of photons in the cavity increases indefinitely in time if and only if $0 < p < 1$. In other words, fluctuations of the atom state are necessary for the pumping to occur. We derive an expression for the mean photon number for any time.

When the leakage is taken into the account, the mean-value of photons number stabilizes in time. The limiting state of the cavity exists and it is independent of the initial state. We study this non-quasi-free non-equilibrium limiting state by computing its characteristic function (see section 2.1.3 for the definition of quasi-free and non-equilibrium steady state). We also calculate the energy flux in both the ideal and open cavity and the entropy production for the ideal cavity.

1.2 Summary of the main results

1.2.1 Lieb-Robinson Bounds and the Existence of the Thermodynamic Limit for the Class of Irreversible Quantum Dynamics

In Chapter 3 we prove Lieb-Robinson bounds for a class of the irreversible quantum dynamics and show the existence of the thermodynamic limit of the dynamics.

We consider a general situation, when the dynamics is described by a Markovian dynamical semigroup. The generator of the dynamics consists of both Hamiltonian and dissipative terms that may depend on time. In the finite volume Λ the generator \mathcal{L} is of the following form

$$\mathcal{L}_\Lambda(t)(A) = \sum_{Z \subset \Lambda} \Psi_Z(t)(A), \quad (1.2.1)$$

$$\begin{aligned} \Psi_Z(t)(A) = & i[\Phi(t, Z), A] \\ & + \sum_{a=1}^{N(Z)} \left(L_a^*(t, Z) A L_a(t, Z) - \frac{1}{2} \{L_a(t, Z)^* L_a(t, Z), A\} \right), \end{aligned}$$

for all local observables A . Here for $Z \subset \Lambda$, $\Phi(t, Z)$ is the local time dependent interaction describing Hamiltonian terms and $L_a(t, Z)$ are local time dependent bounded operators. We put an exponential decay condition in space on the $\Psi_Z(t)$. The detailed set up is discussed in section 3.1.

Fix $T > 0$ and, for all observables A with finite support in Λ , let $A(t), t \in [0, T]$ be a solution of the initial value problem

$$\frac{d}{dt} A(t) = \mathcal{L}_\Lambda(t) A(t), \quad A(0) = A. \quad (1.2.2)$$

For $0 \leq s \leq t \leq T$, define the family of maps $\{\gamma_{t,s}^\Lambda\}_{0 \leq s \leq t} \subset \mathcal{B}(\mathcal{A}_\Lambda, \mathcal{A}_\Lambda)$ by $\gamma_{t,s}^\Lambda(A) = A(t)$, where $A(t)$ is the unique solution of (1.2.2) for $t \in [s, T]$ with initial condition $A(s) = A$. In section 3.2 we show that the dynamics γ_t^Λ generated by (1.2.1) exists and forms a norm-continuous cocycle of unit preserving completely positive maps.

In section 3.3 we prove the Lieb-Robinson bound of the form

$$\|\mathcal{K}\gamma_t^\Lambda(B)\| \leq C(\mathcal{K}, B)e^{-\mu(d(X,Y)-v|t|)}, \quad (1.2.3)$$

where \mathcal{K} is a completely bounded linear operator on the space of observables with support X that vanishes on $\mathbb{1}$ (see section 3.1 for the definition of the completely bounded map), B is an observable with support Y , $d(X, Y)$ is the distance between the supports X and Y , $C(\mathcal{K}, B)$ is constant that depends on \mathcal{K} and B and v is a Lieb-Robinson velocity. It is important that the operator \mathcal{K} can be taken in a form $\mathcal{K}(B) = [A, B]$, which makes the original Lieb-Robinson bound for reversible dynamics (1.1.1) a particular case of the Lieb-Robinson bound in the general form (1.2.3).

In section 3.4 we look at the dynamics γ_t^Λ when the set Λ becomes infinitely large. We prove that the thermodynamic limit of the dynamics exists in the sense of strongly continuous one-parameter cocycle of completely positive unit preserving maps.

1.2.2 Non-equilibrium state of a leaking photon cavity pumped by a random atomic beam

The system we consider consists of the one-mode microwave cavity and a beam of randomly excited atoms. The cavity is described by quantum harmonic oscillator

with the Hamiltonian

$$H_C = \epsilon b^* b,$$

where $\epsilon > 0$ and b^*, b are boson creation and annihilation operators (see section 2.1.1 for more details on these operators).

The beam of atoms is described as a quantum spin chain

$$H_A = \sum_{n \geq 1} E \eta_n,$$

where $E > 0$ and for any $n \geq 1$, the atomic operator $\eta_n := (\sigma^z + \mathbb{1})/2$, where σ^z is the third Pauli matrix.

Atoms are randomly excited with some probability p . They enter the cavity successively and there is only one atom present in the cavity at any given time. Atoms in the excited state interact with the cavity and produce a quantum field. The interaction between the n -th atom and the cavity is the following

$$K(t) = \chi_{[(n-1)\tau, n\tau]}(t)(\lambda(b^* + b) \otimes \eta_n),$$

here $\chi_I(x)$ is a characteristic function of a set I , which is here to make sure that the n -th atom interacts with the cavity only when it is present there, which happen during the time interval $[(n-1)\tau, n\tau]$. The detailed set up of the system is given in the section 4.1.

At first we assume that the cavity insulates an atom-photon system from decohering interactions with its environment. So the Hamiltonian for the system is the

sum of the Hamiltonian of the cavity and atoms and the interaction between them

$$H(t) = \epsilon b^* b \otimes \mathbb{1} + \sum_{n \geq 1} \mathbb{1} \otimes E \eta_n + \sum_{n \geq 1} \chi_{[(n-1)\tau, n\tau]}(t) (\lambda(b^* + b) \otimes \eta_n).$$

In section 4.2 we show that in this case the expected number of photons in the cavity oscillates around a linearly increasing in time function and only a beam of randomly exited atoms ($0 < p < 1$) is able to produce a pumping of the cavity by photons.

In a more general situation we should not assume that the cavity is perfectly insulated. Photons may either escape the cavity or be absorbed into the cavity walls at some constant non-zero rate $\sigma > 0$.

In this case the evolution of the system is described by a Markovian dynamics with the following generator

$$L(t)(\rho_S) = -i[H(t), \rho_S] + \sigma b(\rho_S) b^* - \frac{\sigma}{2} \{b^* b, \rho_S\},$$

where ρ_S is a state of the system and $\sigma > 0$. Here for any observables A and B , $\{A, B\} := AB + BA$ denotes the anticommutator.

In section 4.3 we show that the expected number of photons in the cavity stabilizes in time, which means that the energy leaking out of the cavity equals to the energy coming into the cavity.

In section 4.4 we compute the characteristic function of the limiting state of the cavity on the Weyl algebra (see Section 2.1.2). The limiting state is an infinite convex combination of quasi-free states, and, in general, is not quasi-free itself (see Section 2.1.3 for the definition of a quasi-free state). The limiting state of the system is a

non-equilibrium state in a sense that it is not in a form

$$\omega_\beta(\cdot) = \frac{\text{Tr}(e^{-\beta b^* b} \cdot)}{\text{Tr}(e^{-\beta b^* b})}, \quad (1.2.4)$$

for any $\beta \in \mathbb{C}$.

Section 4.5 is devoted to the calculation of the energy flow for the perfect cavity (4.5.1) and the leaking cavity (4.5.2). The entropy production for the ideal cavity is presented in Section (4.5.3).

Chapter 2

Preliminaries

2.1 Quantum systems

2.1.1 Fock space

The quantum-mechanical states of a particle form a complex Hilbert space \mathcal{H} , for example it could be $L^2(\mathbb{R}^d)$. The Hilbert space of n identical but distinguishable particles is the tensor product

$$\mathcal{H}^n = \bigotimes_{i=1}^n \mathcal{H}.$$

If the number of particles is not fixed, the states are described by vectors in the Fock space $\mathcal{F}(\mathcal{H})$ which is given by

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n \geq 0} \mathcal{H}^n,$$

where $\mathcal{H}^0 = \mathbb{C}$. Therefore a vector $\psi \in \mathcal{F}(\mathcal{H})$ is a sequence of vectors $\psi = \{\psi^{(n)}\}_{n \geq 0}$, where $\psi^{(n)} = f_1 \otimes \dots \otimes f_n$, $f_k \in \mathcal{H}$ for $1 \leq k \leq n$. In quantum physics identical particles are indistinguishable which is reflected by the symmetry under the interchange of the

particle coordinates.

The first case is when the components $\psi^{(n)}$ of each ψ are symmetric under the interchange of coordinates. Such particles are called *bosons*, they form *the Bose-Fock space* $\mathcal{F}_+(\mathcal{H})$. The second case is when the components $\psi^{(n)}$ of each ψ are anti-symmetric under interchange of each pair of coordinates. Such particles are called *fermions* and they form *the Fermi-Fock space* $\mathcal{F}_-(\mathcal{H})$.

Bose- and Fermi-Fock spaces are defined with the help of the operators P_+ and P_- which are defined on $\mathcal{F}(\mathcal{H})$ as follows, for all $f_1, \dots, f_n \in \mathcal{H}$

$$P_+(f_1 \otimes \dots \otimes f_n) = (n!)^{-1} \sum_{\pi} f_{\pi_1} \otimes \dots \otimes f_{\pi_n},$$

$$P_-(f_1 \otimes \dots \otimes f_n) = (n!)^{-1} \sum_{\pi} \epsilon_{\pi} f_{\pi_1} \otimes \dots \otimes f_{\pi_n},$$

where ϵ_{π} is the sign on the permutation π . The sum is taken over all permutations $\pi : (1, 2, \dots, n) \rightarrow (\pi_1, \pi_2, \dots, \pi_n)$. These operator are extended by linearity to two densely defined operators with $\|P_{\pm}\| = 1$. Then extending P_{\pm} by continuity one gets two bounded operators of norm one. The Bose- and Fermi-Fock spaces are given by

$$\mathcal{F}_{\pm}(\mathcal{H}) = P_{\pm}\mathcal{F}(\mathcal{H}).$$

A number operator N is defined on $\mathcal{F}(\mathcal{H})$ by

$$N\psi = \{n\psi^{(n)}\}_{n \geq 0}$$

with the domain $D(N) = \{\psi = \{\psi^{(n)}\}_{n \geq 0} : \sum_{n \geq 0} n^2 \|\psi^{(n)}\|^2 < \infty\}$.

The particle creation and annihilation operators on $\mathcal{F}(\mathcal{H})$ are defined as follows.

For each $f \in \mathcal{H}$, $a(f)\psi^{(0)} = 0$ and $a^*(f)\psi^{(0)} = f$ and

$$\begin{aligned} a_{\mathcal{F}}(f)(f_1 \otimes \dots \otimes f_n) &= \sqrt{n}(f, f_1)f_2 \otimes \dots \otimes f_n, \\ a_{\mathcal{F}}^*(f)(f_1 \otimes \dots \otimes f_n) &= \sqrt{n+1}f \otimes f_1 \otimes \dots \otimes f_n, \end{aligned}$$

for any $f_k \in \mathcal{H}$, $k \in \mathbb{Z}$. Extending by linearity one gets two densely defined operators.

If $\psi^{(n)} \in \mathcal{H}^n$ we get

$$\|a_{\mathcal{F}}(f)\psi^n\| \leq \sqrt{n}\|f\|\|\psi^n\|, \quad \|a_{\mathcal{F}}^*(f)\psi^n\| \leq \sqrt{n+1}\|f\|\|\psi^n\|.$$

Therefore $a_{\mathcal{F}}(f)$ and $a_{\mathcal{F}}^*(f)$ have well-defined extensions to the domain of $D(N^{1/2})$ of $N^{1/2}$.

From the definition one can see that a^* operator is indeed the adjoint of a . For all $\phi, \psi \in D(N^{1/2})$ and $f \in \mathcal{H}$ we have that

$$(a_{\mathcal{F}}^*(f)\phi, \psi) = (\phi, a_{\mathcal{F}}(f)\psi).$$

The creation and annihilation operators on the Fock spaces $\mathcal{F}_{\pm}(\mathcal{H})$ are defined by

$$\begin{aligned} a_{\pm}(f) &= P_{\pm}a(f)P_{\pm} = a(f)P_{\pm}, \\ a_{\pm}^*(f) &= P_{\pm}a^*(f)P_{\pm} = P_{\pm}a^*(f). \end{aligned}$$

In future when it is clear what Fock space we are working in we may drop the indices \pm . In Section 4 we will use one particle boson creation and annihilation operators b^*, b that act on the Hilbert space \mathcal{H} .

The important properties of creation and annihilation operators are canonical

commutation relations (CCR) for boson creation and annihilation operators and canonical anti-commutation relations (CAR) for fermion creation and annihilation operators.

Boson operators satisfy the following CCR relations

$$\begin{aligned} [a_+(f), a_+(g)] &= 0 = [a_+^*(f), a_+^*(g)], \\ [a_+(f), a_+^*(g)] &= (f, g)\mathbb{1}. \end{aligned} \tag{2.1.1}$$

Fermion operators satisfy the following CAR relations

$$\begin{aligned} \{a_-(f), a_-(g)\} &= 0 = \{a_-^*(f), a_-^*(g)\} \\ \{a_-(f), a_-^*(g)\} &= (f, g)\mathbb{1}. \end{aligned} \tag{2.1.2}$$

2.1.2 Weyl operators

In this section we will consider the Boson-Fock space $\mathcal{F}_+(\mathcal{H})$. So we will drop the plus index on the creation, annihilation operators.

It is convenient to introduce the family of operators $\{b(f); f \in \mathcal{H}\}$ as follows

$$b(f) = a(f) + a^*(f),$$

where a^*, a are boson creation and annihilation operators. Each $b(f)$ is a self-adjoint linear operator on the Fock space. The canonical commutation relations (2.1.1) in terms of the operators b now take form

$$[b(f), b(g)] = 2i \operatorname{Im} (f, g).$$

The Weyl operator is defined as follows

$$W(f) = \exp\{ib(f)\}.$$

The operators $b(f)$ as well as the creation and annihilation operators are unbounded, while the Weyl operator is unitary. This is one of the reasons it is easier to work with Weyl operators than with creation and annihilation operators.

The Baker-Campbell-Hausdorff formula shows that for two operators X and Y

$$e^Xe^Y = e^{X+Y}e^{\frac{1}{2}[X,Y]},$$

if $[X, [X, Y]] = 0 = [Y, [X, Y]]$. Using this formula one can get the Weyl form of the canonical commutation relations

$$W(f)W(g) = e^{-\frac{i}{2}\text{Im}(f,g)}W(f+g) = e^{-i\text{Im}(f,g)}W(g)W(f).$$

In Section 4 the creation and annihilation operators are b^* and b that act on the Hilbert space \mathcal{H} . The Weyl operator is then defined as follows: for every $\alpha \in \mathbb{C}$,

$$W(\alpha) = \exp\{\alpha b - \bar{\alpha} b^*\}. \quad (2.1.3)$$

From Baker-Cambell-Hausdorff formula we obtain the following relationship

$$W(\alpha) = e^{\alpha b - \bar{\alpha} b^*} = e^{-\bar{\alpha} b^*} e^{\alpha b} e^{-\frac{|\alpha|^2}{2}}. \quad (2.1.4)$$

From canonical commutation relations (2.1.1) we also find that for any $\beta, \gamma \in \mathbb{C}$

$$e^{\beta b^*} b = (b - \beta) e^{\beta b^*} \text{ and } e^{\gamma b} b^* = (b^* + \gamma) e^{\gamma b}. \quad (2.1.5)$$

Denote the Weyl CCR algebra of observables generated by all the Weyl operators as

$$\mathcal{U}(\mathcal{H}) := \{W(\alpha) : \alpha \in \mathbb{C}\}.$$

The linear map $T : \mathcal{U}(\mathcal{H}) \rightarrow \mathcal{U}(\mathcal{H})$ is called *quasi-free*, if it has the form:

$$T(W(f)) = \Psi(f)W(\Gamma(f)) \quad f \in \mathcal{H}, \quad (2.1.6)$$

for some linear map $\Gamma : \mathcal{H} \rightarrow \mathcal{H}$ and complex function $\Psi : \mathcal{H} \rightarrow \mathbb{C}$.

2.1.3 States

The *state* ω on the algebra of observables \mathcal{U} maps each observable A into its expectation value, which is in general a complex number $\omega(A)$ with the properties

- 1) normalization: $\omega(\mathbb{1}) = 1$
- 2) linearity: for each pair A, B of observables and each pair λ, μ of complex numbers one has $\omega(\lambda A + \mu B) = \lambda \omega(A) + \mu \omega(B)$
- 3) positivity: for each observable A , $\omega(A^* A) \geq 0$.

Let ω be any state on the Weyl algebra $\mathcal{U}(\mathcal{H})$. This state is well-defined if for all $f \in \mathcal{H}$ all the expectation values $\omega(W(f))$ are known, in other words it is completely defined by its characteristic functional:

$$\mathcal{H} \ni f \mapsto \omega(W(f)).$$

We consider states such that their characteristic functional could be written in terms of correlation functions

$$\begin{aligned}\omega(W(f)) &= \omega(e^{ib(f)}) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \omega(b(f)^n) \\ &= \exp\left\{\sum_{n=1}^{\infty} \frac{i^n}{n!} \omega(b(f)^n)_t\right\},\end{aligned}\tag{2.1.7}$$

where the truncated correlation functions $\omega(\dots)_t$ are defined recursively

$$\omega(b(f_1)\dots b(f_n)) = \sum \omega(b(f_1)\dots)_t \dots \omega(\dots b(f_n))_t,$$

here the sum is taken over all possible ordered partitions $(1, \dots), \dots (\dots, n)$ of the set $\{1, \dots, n\}$. The expression $\omega(b(f_1)\dots b(f_n))_t$ is called the truncated correlation function of order n . See [22] for more details.

A state ω of the boson algebra of observables is called a *quasi-free state*, if all its truncated correlation functions of orders $n > 2$ vanish.

From (2.1.7) it follows that the general quasi-free state is determined by its one- and two-point correlation functions and therefore it is of the following form

$$\begin{aligned}\omega(W(f)) &= \exp\left\{i\omega(b(f)) - \frac{1}{2}\omega(b(f)b(f))_t\right\} \\ &= \exp\left\{i\omega(b(f)) - \frac{1}{2}(\omega(b(f)^2) - \omega(b(f))^2)\right\}\end{aligned}\tag{2.1.8}$$

$$= \exp\left\{ir(f) - \frac{1}{2}s(f, f)\right\},\tag{2.1.9}$$

where we denoted $r(f) = \omega(b(f))$ and $s(f, f) = \omega(b(f)^2) - \omega(b(f))^2$.

A state ω is called *regular* if the function $a \mapsto \omega(W(af))$ is continuous for all $f \in \mathcal{H}$ and every $a \in \mathbb{C}$. Characteristic functionals of regular states on $\text{CCR}(\mathfrak{H})$ are

characterized by Araki and Segal [1, 2] in the following theorem.

Theorem 1. *A map $\mathcal{H} \ni f \mapsto \omega(W(f)) \in \mathbb{C}$ is the characteristic functional of a regular state ω on $CCR(\mathcal{H})$ if and only if*

1. $\omega(W(0)) = 1$.
2. *The function $a \mapsto \omega(W(af))$ is continuous for all $f \in \mathcal{H}$, $a \in \mathbb{C}$.*
3. *For any integer $n \geq 2$, all $f_1, \dots, f_n \in \mathcal{H}$ and all $z_1, \dots, z_n \in \mathbb{C}$ one has*

$$\sum_{j,k=1}^n \omega(W(f_j - f_k)) e^{-i \operatorname{Im}(f_j, f_k)/2} \overline{z_j} z_k \geq 0.$$

The Gibbs states or *the equilibrium states* are given by

$$\omega_\beta(A) = \frac{\operatorname{Tr}(e^{-\beta a^* a} A)}{\operatorname{Tr}(e^{-\beta a^* a})}, \quad (2.1.10)$$

for any $\beta \in \mathbb{C}$. The state that is not in this form is called a *non-equilibrium state*. It can be shown that every Gibbs state of the form (2.1.10) is quasi-free (see [22] for details).

2.2 Dynamics of quantum systems

2.2.1 Reversible dynamics

In quantum mechanics for a physical system described in terms of a Hilbert space \mathcal{H} a state $\phi \in \mathcal{H}$ of the systems evolves in time under the action of a one-parameter strongly continuous group of unitary operators.

A family of unitary maps $U_t \in \mathcal{B}(\mathcal{H})$, $t \in \mathbb{R}$ is called a *strongly continuous one-parameter group* of unitary operators if

- 1) $U_t U_s = U_{t+s}$, for $s, t \in \mathbb{R}$,
- 2) $\lim_{t \rightarrow 0} \|U_t \phi - \phi\| = 0$, for $\phi \in \mathcal{H}$,
- 3) $U_t^{-1} = U_{-t}$.

If $\{U_t\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter group of unitary operators, then by Stone's theorem [] there exists a self-adjoint operator H , the Hamiltonian of the system, such that

$$U_t = e^{-itH},$$

for $t \in \mathbb{R}$.

The dynamics of the system can be described using two equivalent perspectives. Either a state changes in time while all observables remain the same, or the dynamical group acts on the set of observables leaving the states unchanged.

The second case is called "the Heisenberg picture". The time evolution of the observable $A \in \mathcal{B}(\mathcal{H})$ is described by

$$\tau_t(A) = U_t^* A U_t = e^{itH} A e^{-itH}.$$

2.2.2 Lieb-Robinson bounds for reversible dynamics

For Lieb-Robinson bounds the quantum systems are considered on the countable set of vertices Γ (called lattice) which is equipped with a metric d . A Hilbert space \mathcal{H}_x is assigned to each vertex $x \in \Gamma$. For any finite subset $\Lambda \subset \Gamma$ the Hilbert space

associated with it is the tensor product

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x. \quad (2.2.1)$$

The local algebra of observables over Λ is

$$\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{B}(\mathcal{H}_x),$$

where $\mathcal{B}(\mathcal{H}_x)$ denotes the algebra of bounded linear operators on \mathcal{H}_x .

If $\Lambda_1 \subset \Lambda_2$, then we may identify \mathcal{A}_{Λ_1} in a natural way with the subalgebra $\mathcal{A}_{\Lambda_1} \otimes \mathbb{1}_{\Lambda_2 \setminus \Lambda_1}$ of \mathcal{A}_{Λ_2} , and simply write $\mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2}$. Then the algebra of local observables is defined as an inductive limit

$$\mathcal{A}_\Gamma^{\text{loc}} = \bigcup_{\Lambda \subset \Gamma} \mathcal{A}_\Lambda. \quad (2.2.2)$$

See [] for more detailed on the inductive limit. The C^* -algebra of quasi-local observables \mathcal{A}_Γ is the norm completion of $\mathcal{A}_\Gamma^{\text{loc}}$.

The *support* of the observable $A \in \mathcal{A}_\Lambda$ is the minimal set $X \subset \Lambda$ for which $A = A' \otimes \mathbb{1}_{\Lambda \setminus X}$ for some $A' \in \mathcal{A}_X$.

We assume that there exists a non-increasing function $F : [0, \infty) \rightarrow (0, \infty)$ such that:

i) F is uniformly integrable over Γ , i.e.,

$$\|F\| := \sup_{x \in \Gamma} \sum_{y \in \Gamma} F(d(x, y)) < \infty, \quad (2.2.3)$$

and

ii) F satisfies

$$C := \sup_{x,y \in \Gamma} \sum_{z \in \Gamma} \frac{F(d(x,z))F(d(y,z))}{F(d(x,y))} < \infty. \quad (2.2.4)$$

Example 1. As an example one may take $\Gamma = \mathbb{Z}^\nu$ for some integer $\nu \geq 1$ with the metric $d(x,y) = |x - y| = \sum_{j=1}^\nu |x_j - y_j|$. In this case, the function F can be chosen as $F(|x|) = (1 + |x|)^{-\nu-\epsilon}$ for any $\epsilon > 0$. To show that the constant C is finite we use the triangle inequality and the symmetry of the taken supremum over x and y

$$\begin{aligned} C &= \sup_{x,y \in \mathbb{Z}^\nu} \sum_{z \in \mathbb{Z}^\nu} \frac{(1 + |x - y|)^{\nu+\epsilon}}{(1 + |x - z|)^{\nu+\epsilon}(1 + |y - z|)^{\nu+\epsilon}} \\ &\leq \sup_{x,y \in \mathbb{Z}^\nu} \sum_{z \in \mathbb{Z}^\nu} \frac{(1 + |x - z| + |y - z| + 1)^{\nu+\epsilon}}{(1 + |x - z|)^{\nu+\epsilon}(1 + |y - z|)^{\nu+\epsilon}} \\ &\leq \sup_{x,y \in \mathbb{Z}^\nu} \sum_{z \in \mathbb{Z}^\nu} \left(\frac{1}{1 + |x - z|} + \frac{1}{1 + |y - z|} \right)^{\nu+\epsilon}. \end{aligned}$$

Now we use the inequality between geometric and algebraic mean: for any $\alpha \geq 1$ and any $a, b \geq 0$,

$$\left(\frac{a + b}{2} \right)^\alpha \leq \frac{a^\alpha + b^\alpha}{2}.$$

Continuing the calculations

$$\begin{aligned} C &\leq \sup_{x,y \in \mathbb{Z}^\nu} \sum_{z \in \mathbb{Z}^\nu} 2^{\nu+\epsilon-1} \left(\left(\frac{1}{1 + |x - z|} \right)^{\nu+\epsilon} + \left(\frac{1}{1 + |y - z|} \right)^{\nu+\epsilon} \right) \\ &= 2^{\nu+\epsilon-1} \left(\sup_{x \in \mathbb{Z}^\nu} \sum_{z \in \mathbb{Z}^\nu} \left(\frac{1}{1 + |x - z|} \right)^{\nu+\epsilon} + \sup_{y \in \mathbb{Z}^\nu} \sum_{z \in \mathbb{Z}^\nu} \left(\frac{1}{1 + |y - z|} \right)^{\nu+\epsilon} \right). \end{aligned}$$

Since two supremums are the same and each of them is achieved at any value we

have

$$\begin{aligned} C &\leq 2^{\nu+\epsilon} \sup_{x \in \mathbb{Z}^\nu} \sum_{z \in \mathbb{Z}^\nu} \left(\frac{1}{1 + |x - z|} \right)^{\nu+\epsilon} \\ &\leq 2^{\nu+\epsilon} \sum_{w \in \mathbb{Z}^\nu} \frac{1}{(1 + |w|)^{\nu+\epsilon}} < \infty. \end{aligned}$$

Having a set Γ with a function F that satisfies (2.2.3) and (2.2.4), we can define for any $\mu > 0$ the function

$$F_\mu(d) = e^{-\mu d} F(d), \quad (2.2.5)$$

which then also satisfies i) and ii) with $\|F_\mu\| \leq \|F\|$ and $C_\mu \leq C$.

The Hamiltonian of the system is described by local Hamiltonians $H^{loc} = \{H_x\}$, where H_x is a not necessarily bounded self-adjoint operator over \mathcal{H}_x , and bounded perturbations defined in terms of interaction Φ . The interaction Φ is a map from the set of subsets of Γ to \mathcal{A}_Γ with the properties that for each finite set $X \subset \Gamma$, $\Phi(X) \in \mathcal{A}_X$ and $\Phi(X)^* = \Phi(X)$. For any $\mu \geq 0$, denote by $\mathcal{B}_\mu(\Gamma)$ the set of interactions for which

$$\|\Phi\|_\mu = \sup_{x, y \in \Gamma} \sum_{X \ni x, y} \frac{\|\Phi(X)\|}{F_\mu(x, y)} < \infty. \quad (2.2.6)$$

For every finite subset $\Lambda \subset \Gamma$ the Hamiltonian is of the form

$$H_\Lambda = H_\Lambda^{loc} + H_\Lambda^\Phi = \sum_{x \in \Lambda} H_x + \sum_{X \subset \Lambda} \Phi(X).$$

Since these operators are self-adjoint they generate a Heisenberg dynamics

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}$$

for any $A \in \mathcal{A}_\Lambda$.

The following theorem provides the Lieb-Robinson bounds.

Theorem 2. *Let X and Y be two disjoint subsets of Λ . Then for any pair of local observables $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ one has that*

$$\|[\tau_t^\Lambda(A), B]\| \leq \frac{2\|A\|\|B\|}{C_\mu} (e^{2\|\Phi\|_\mu C_\mu |t|} - 1) \sum_{x \in X} \sum_{y \in Y} F_\mu(d(x, y)). \quad (2.2.7)$$

The bound in the theorem could be rewritten using the properties of the function F as follows

$$\|[\tau_t^\Lambda(A), B]\| \leq \frac{2\|A\|\|B\|}{C_\mu} \|F\| \min(|X|, |Y|) e^{-\mu(d(X, Y) - v_\mu |t|)}, \quad (2.2.8)$$

where $v_\mu = \frac{2\|\Phi\|_\mu C_\mu}{\mu}$ is the Lieb-Robinson velocity.

This theorem is a special case of the general Lieb-Robinson bound (1.2.3) that we are going to consider here, so the proof of this theorem is included in the proof in section 3.3 as a particular case and it also can be found separately in [17].

2.2.3 Existence of the dynamics in the thermodynamic limit

We assume that the set Γ , which is still equipped with a metric d , as a countable set with infinite cardinality. One can take, for example, $\Gamma = \mathbb{Z}^\nu$, for some $\nu \geq 1$.

The thermodynamic limit is taken over an increasing exhausting sequence of finite subsets $\Lambda_n \subset \Gamma$.

Theorem 3. *Let $\mu > 0$ and $\Phi \in \mathcal{B}_\mu(\Gamma)$. The dynamics τ_t corresponding to Φ exists as a strongly continuous one-parameter group of automorphisms on \mathcal{A}_Γ such that for*

all $t \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} \|\tau_t^\Lambda(A) - \tau_t(A)\| = 0$$

for all $A \in \mathcal{A}_\Gamma$.

The proof of this theorem can be found in [17] or, as a particular case, in section 3.4 where we prove the existence of the thermodynamic limit for the irreversible dynamics.

2.2.4 Irreversible dynamics

In general, if we consider an open system taking into the account the interaction between the system and an environment we have to consider a non-Hamiltonian system. The dynamical maps γ_t of such system form a *one-parameter completely positive dynamical semigroup* on an algebra of observables \mathcal{U} , which is described by the properties

- 1) γ_t is completely positive,
- 2) $\gamma_t(I) = I$,
- 3) $\gamma_s \gamma_t = \gamma_{t+s}$,
- 4) $\gamma_t \rightarrow \mathbb{1}$ when $t \rightarrow 0$ strongly, i.e. for any $A \in \mathcal{U}$, $\lim_{t \rightarrow 0} \|\gamma_t(A) - A\| = 0$.

A map γ is called *completely positive* if for any $n > 1$, the linear maps $\gamma \otimes \text{id}_{M_n}$ where M_n are $n \times n$ complex matrices, are positive.

A strongly continuous group of automorphisms is a special case of a dynamical semigroup for $\gamma_t(A) = \tau_t(A)$.

The dynamics γ_t of the open system is the solution to the *Markovian master equation*

$$\frac{d}{dt} \gamma_t(A) = \mathcal{L}(\gamma_t(A)), \quad (2.2.9)$$

for all $t \in \mathbb{R}$, with $\gamma_0(A) = A$, where the generator \mathcal{L} acts on the space of observables.

It was shown in [13] that the generator \mathcal{L} is in the Lindblad-Kossakowski form

$$\mathcal{L}(A) = i[H, A] + \sum_a (L_a^* A L_a - \frac{1}{2} \{L_a^* L_a, A\}), \quad (2.2.10)$$

where L_a are such that $\sum_a L_a^* L_a$ is a bounded operator and here $\{A, B\} = AB + BA$ is the anti-commutator. The Hamiltonian H have bounded interaction terms, but may have unbounded on-site terms as in section 2.2.2, making the generator \mathcal{L} unbounded.

The proof that the time evolution of an open system is Markovian in the weak coupling limit can be found in [6]. It was shown there that the limit of the unitary dynamics that depends on a coupling constant may not be unitary, but it satisfies the Markovian master equation (2.2.9) when the coupling constant converges to zero for a rescaled time variable.

In this dissertation we are going to discuss the dynamics generated by the observables of the type (2.2.10), where H and each observable L_a are time-dependent. We will show that the evolution with this type of generator is well-defined as a norm-continuous cocycle of unit preserving completely positive maps.

2.2.5 Quasi-free dynamics

The unity-preserving *quasi-free semigroup* $\{T_t\}_{t \geq 0}$ on the CCR algebra $\mathcal{U}(\mathcal{H})$ is defined in a similar way:

$$T_t(W(f)) := \Psi_t(f) W(\Gamma_t(f)) , \quad f \in \mathcal{H} , \quad (2.2.11)$$

where $\Psi_{t=0}(f) = 1$, $\Gamma_{t=0} = \mathbb{1}$ and $\Psi_t(f = 0) = 1$, $\Gamma_t(f = 0) = 0$. Note that the semigroup property of $\{T_t\}_{t \geq 0}$ and (2.2.11) imply

$$\begin{aligned} T_{s+t}(W(f)) &= T_s(T_t(W(f))) \\ &= \Psi_s(\Gamma_t(f)) \Psi_t(f) W(\Gamma_s(\Gamma_t(f))) = \Psi_{s+t}(f) W(\Gamma_{s+t}(f)) . \end{aligned} \quad (2.2.12)$$

Then linear independence of the Weyl operators yields

$$\Gamma_{s+t}(f) = \Gamma_s(\Gamma_t(f)) \quad \text{and} \quad \Psi_{s+t}(f) = \Psi_s(\Gamma_t(f)) \Psi_t(f) .$$

Hence, $\{\Gamma_t\}_{t \geq 0}$ is in turn a semigroup on \mathcal{H} .

Note that by definitions (2.2.11) and (2.1.8) the quasi-free dynamics maps the quasi-free states into the states:

$$\omega_{r,s}(T_t(W(f))) = \Psi_t(f) \omega_{r,s}(W(\Gamma_t(f))) = \Psi_t(f) \omega_{r_t, s_t}(W(f)) , \quad (2.2.13)$$

where $r_t(f) := r(\Gamma_t(f))$ and $s_t(f, f) := s(\Gamma_t(f), \Gamma_t(f))$. In general the states (2.2.13) are not quasi-free.

Chapter 3

Lieb-Robinson bounds and the existence of the thermodynamic limit for a class of irreversible quantum dynamics

3.1 Set up

The set up here is the similar to the set up for the Lieb-Robinson bounds for unitary dynamics as in section 2.2.2. The system is considered on the countable set Γ equipped with a metric d . We put the same restriction on the lattice: there is a non-increasing function $F : [0, \infty] \rightarrow (0, \infty)$ that satisfies (2.2.3) and (2.2.4).

The Hilbert space of states \mathcal{H}_Λ of any finite subsystem $\Lambda \subset \Gamma$ is defined by (2.2.1) as a tensor product of Hilbert spaces \mathcal{H}_x of every point x in Λ . The C^* -algebra of quasi-local observables \mathcal{A}_Γ is the norm completion of the algebra of local observables

\mathcal{A}_Γ^{loc} defined in (2.2.2) as a space of bounded linear operators on \mathcal{H}_Λ .

The generator of the dynamics is now consists of two parts: the Hamiltonian interactions and the dissipative terms. We allow both of them to be time-dependent.

As in the case of unitary dynamics, the Hamiltonian part is described by an interaction $\Phi(t, \cdot)$, which is now time-dependent. For every time $t \in \mathbb{R}$ the interaction $\Phi(t, \cdot)$ is a map from the set of finite subsets of Γ to \mathcal{A}_Γ , such that for every finite $Z \subset \Gamma$

1. $\Phi(t, Z) \in \mathcal{A}_Z$,
2. $\Phi(t, Z)^* = \Phi(t, Z)$.

The dissipative part is described by terms of Lindblad form. For every finite $Z \subset \Gamma$ these terms are defined by a set of operators $L_a(t, Z) \in \mathcal{A}_Z$, where $a = 1, \dots, N(Z)$. It is possible for $N(Z) = \infty$ with an additional assumption of the convergence of the sum.

Then, for any finite set $\Lambda \subset \Gamma$ and time $t \in \mathbb{R}$ the generator $\mathcal{L}_\Lambda \in \mathcal{B}(\mathcal{A}_\Lambda, \mathcal{A}_\Lambda)$ is defined as follows: for all $A \in \mathcal{A}_\Lambda$,

$$\mathcal{L}_\Lambda(t)(A) = \sum_{Z \subset \Lambda} \Psi_Z(t)(A), \text{ where} \quad (3.1.1)$$

$$\begin{aligned} \Psi_Z(t)(A) = & i[\Phi(t, Z), A] \\ & + \sum_{a=1}^{N(Z)} \left(L_a^*(t, Z) A L_a(t, Z) - \frac{1}{2} \{L_a(t, Z)^* L_a(t, Z), A\} \right), \end{aligned} \quad (3.1.2)$$

here $\{A, B\} = AB + BA$ is the anticommutator of A and B . The operator $\Psi_Z(t)$ can be viewed as a bounded linear transformation on \mathcal{A}_Λ for any $\Lambda \supset Z$, which can be written in the form $\Psi_Z(t) \otimes \text{id}_{\mathcal{A}_{\Lambda \setminus Z}}$. For every $Z \subset \Lambda$ the norm of these maps can be

bounded independently of the choice of $\Lambda \supset Z$ as follows:

$$\|\Psi_Z(t)\| \leq 2\|\Phi(t, Z)\| + 2 \sum_{a=1}^{N(Z)} \|L_a(t, Z)\|^2.$$

If $N(Z) = \infty$ we can insure that the sums $\sum_{a=1}^{N(Z)} \|L_a(t, Z)\|^2$ converge guaranteeing the uniform boundedness of the maps $\Psi_Z(t)$. However, it is more general to assume that the maps $\Psi_Z(t)$ defined on \mathcal{A}_Z are completely bounded. A map $\Psi \in \mathcal{B}(\mathcal{A}_Z)$ is called *completely bounded* if for all $n > 1$, the linear maps $\Psi \otimes \text{id}_{M_n}$, where $M_n = \mathcal{B}(\mathbb{C}^n)$ are $n \times n$ complex matrices, defined on $\mathcal{A}_Z \otimes M_n$ are bounded with uniformly bounded norm. The *cb-norm* is then defined by

$$\|\Psi\|_{\text{cb}} = \sup_{n \geq 1} \|\Psi \otimes \text{id}_{M_n}\| < \infty.$$

By this definition the cb-norm of $\Psi_Z(t) \in \mathcal{B}(\mathcal{A}_Z, \mathcal{A}_Z)$ is independent of any Λ such that $Z \subset \Lambda \subset \Gamma$.

The main assumption that we make is the following.

Assumption 1. *Given (Γ, d) and F as described at the beginning of this section, the following hypotheses hold:*

1. *For all finite $\Lambda \subset \Gamma$, $\mathcal{L}_\Lambda(t)$ is norm-continuous in t (with the uniform operator norm on $\mathcal{B}(\mathcal{A}_\Lambda, \mathcal{A}_\Lambda)$), and hence uniformly continuous on compact intervals.*
2. *There exists $\mu > 0$ such that for every $t \in \mathbb{R}$*

$$\|\Psi\|_{t, \mu} := \sup_{s \in [0, t]} \sup_{x, y \in \Lambda \subset \Gamma} \sum_{Z \ni x, y} \frac{\|\Psi_Z(s)\|_{\text{cb}}}{F_\mu(d(x, y))} < \infty. \quad (3.1.3)$$

where $\|\cdot\|_{\text{cb}}$ denotes the cb-norm of completely bounded maps.

The second part of the assumption means that the interaction weakens exponentially as the diameter of the support grows. This condition is similar to (2.2.6) in unitary dynamics case.

From definitions note that we have

$$\|\mathcal{L}_\Lambda(t)\| \leq \sum_{Z \subset \Lambda} \|\Psi_Z(t)\| \leq \sum_{x,y \in \Lambda} \sum_{Z \ni x,y} \|\Psi_Z(t)\|_{\text{cb}} \leq \|\Psi\|_{t,\mu} |\Lambda| \|F\|.$$

Define

$$M_t = \|\Psi\|_{t,\mu} |\Lambda| \|F\|. \quad (3.1.4)$$

From the definition (3.1.3) it is clear that $M_s \leq M_t$ for $s < t$.

To define the dynamics of the system fix $T > 0$ and, for all $A \in \mathcal{A}_\Lambda$, let $A(t), t \in [0, T]$ be a solution of the initial value problem

$$\frac{d}{dt} A(t) = \mathcal{L}_\Lambda(t) A(t), \quad A(0) = A. \quad (3.1.5)$$

Since $\|\mathcal{L}_\Lambda(t)\| \leq M_T < \infty$, this solution exists and is unique by the standard existence and uniqueness results for ordinary differential equations.

For $0 \leq s \leq t \leq T$, define the family of maps $\{\gamma_{t,s}^\Lambda\}_{0 \leq s \leq t} \subset \mathcal{B}(\mathcal{A}_\Lambda, \mathcal{A}_\Lambda)$ by

$$\gamma_{t,s}^\Lambda(A) = A(t),$$

where $A(t)$ is the unique solution of (3.1.5) for $t \in [s, T]$ with initial condition $A(s) = A$.

Then, the *cocycle property*, $\gamma_{t,s}(A(s)) = A(t)$, follows from the uniqueness of the solution of (3.1.5).

Recall that a linear map $\gamma : \mathcal{A} \rightarrow \mathcal{B}$, where \mathcal{A} and \mathcal{B} are C^* -algebras is called

3.2. Existence of a dynamics of the finite system as a semigroup of completely positive unit preserving maps 30

completely positive if the maps $\gamma \otimes \text{id} : \mathcal{A} \otimes M_n \rightarrow \mathcal{B} \otimes M_n$ are positive for all $n \geq 1$. Here M_n stands for the $n \times n$ matrices with complex entries, and positive means that positive elements (i.e., elements of the form A^*A) are mapped into positive elements.

3.2 Existence of a dynamics of the finite system as a semigroup of completely positive unit preserving maps

In this section we show that the dynamics defined in the previous section exists as a norm-continuous cocycle of unit preserving completely positive maps. This extends the well-known result for time-independent generators of Lindblad form [?] to the time-dependent case. The theorem is formulated for more general generators of the Markovian dynamics, which includes the generators of the type (3.1.1).

Theorem 4. *Let \mathcal{A} be a C^* -algebra, $T > 0$, and for $t \in [0, T]$, let $\mathcal{L}(t)$ be a norm-continuous family of bounded linear operators on \mathcal{A} . If*

$$(i) \quad \mathcal{L}(t)(\mathbb{1}) = 0;$$

$$(ii) \quad \text{for all } A \in \mathcal{A}, \mathcal{L}(t)(A^*) = \mathcal{L}(t)(A)^*;$$

$$(iii) \quad \text{for all } A \in \mathcal{A}, \mathcal{L}(t)(A^*A) - \mathcal{L}(t)(A^*)A - A^*\mathcal{L}(t)(A) \geq 0;$$

then the maps $\gamma_{t,s}$, $0 \leq s \leq t \leq T$, defined by equation (3.1.5), are a norm-continuous cocycle of unit preserving completely positive maps.

It is straightforward to check that the $\mathcal{L}_\Lambda(t)$ defined in (3.1.1) satisfy properties (i) and (ii). Property (iii), which is called *complete dissipativity*, follows immediately

from the observation

$$\mathcal{L}_\Lambda(t)(A^*A) - \mathcal{L}_\Lambda(t)(A^*)A - A^*\mathcal{L}_\Lambda(t)(A) = \sum_{Z \subset \Lambda} \sum_{a=1}^{N(Z)} [A, L_a(t, Z)]^* [A, L_a(t, Z)] \geq 0.$$

Therefore, using this result, we conclude that, under Assumption 1, for all finite $\Lambda \subset \Gamma$, the maps $\gamma_{t,s}^\Lambda$, $0 \leq s \leq t$, form a norm-continuous cocycle of completely positive and unit preserving maps.

Before we begin the proof we make the simplification of some notations.

Let $\mathcal{L}(t)$, $t \geq 0$, denote a family of operators on a C^* -algebra \mathcal{A} satisfying the assumptions of Theorem 4 and for $0 \leq s \leq t$ consider the maps $\mathcal{A} \ni A \mapsto \gamma_{t,s}(A)$ defined by the solutions of (3.1.5) with initial condition A at $t = s$. Without loss of generality we can assume $s = 0$ in the proof of the theorem because, if we denote $\tilde{\mathcal{L}}(t) = \mathcal{L}(t+s)$, then $\gamma_{t,s} = \tilde{\gamma}_{t-s,0}$, where $\tilde{\gamma}_{t,0}$ is the maps determined by the generators $\tilde{\mathcal{L}}(t)$.

The maps $\gamma_{t,s}$ satisfy the equation

$$\gamma_{t,s} = \text{id} + \int_s^t \mathcal{L}(\tau) \gamma_{\tau,s} d\tau. \quad (3.2.1)$$

In our proof of the complete positivity of $\gamma_{t,0}$ we will use an expression for $\gamma_{t,0}$ as the limit of an Euler product, i.e. approximations $T_n(t)$ defined by

$$T_n(t) = \prod_{k=n}^1 \left(\text{id} + \frac{t}{n} \mathcal{L}\left(\frac{kt}{n}\right) \right). \quad (3.2.2)$$

The product is taken in the order so that the factor with $k = 1$ is on the right.

Lemma 1. *Let $\mathcal{L}(t)$, $t \geq 0$, denote a family of operators on a C^* -algebra \mathcal{A} satisfying*

the assumptions of Theorem 4. Then, uniformly for all $t \in [0, T]$,

$$\lim_{n \rightarrow \infty} \|T_n(t) - \gamma_{t,0}\| = 0,$$

where $T_n(t)$ is defined by (3.2.2).

Proof. From the cocycle property established in Section 3.1, we have

$$\gamma_{t,0} = \prod_{k=n}^1 \gamma_{t \frac{k}{n}, t \frac{k-1}{n}}.$$

Now, consider the difference

$$\begin{aligned} T_n(t) - \gamma_{t,0} &= \prod_{k=n}^1 \left(\text{id} + \frac{t}{n} \mathcal{L}\left(\frac{kt}{n}\right) \right) - \prod_{k=n}^1 \gamma_{t \frac{k}{n}, t \frac{k-1}{n}} \\ &= \sum_{j=1}^n \left[\prod_{k=n}^{j+1} \left(\text{id} + \frac{t}{n} \mathcal{L}\left(t \frac{k-1}{n}\right) \right) \right] \left[\left(\text{id} + \frac{t}{n} \mathcal{L}\left(t \frac{j-1}{n}\right) \right) - \gamma_{t \frac{j}{n}, t \frac{j-1}{n}} \right] \gamma_{t \frac{j-1}{n}, 0}. \end{aligned}$$

To estimate the norm of this difference we look at each factor separately.

Using the boundedness of $\mathcal{L}(t)$ and the fact that M_t , defined in (3.1.4), is increasing in t , the norm of the first factor is bounded from above by

$$\left\| \prod_{k=n}^{j+1} \left(\text{id} + \frac{t}{n} \mathcal{L}\left(t \frac{k-1}{n}\right) \right) \right\| \leq \prod_{k=n}^1 \left(1 + \frac{t}{n} \|\mathcal{L}(t \frac{k-1}{n})\| \right) \leq \left(1 + \frac{t}{n} M_t \right)^n.$$

To bound the second factor notice that from (3.2.1) we obtain

$$\|\gamma_{t,s}\| \leq 1 + \int_s^t \|\mathcal{L}(\tau)\| \|\gamma_{\tau,s}\| d\tau.$$

Then by Gronwall inequality [11, Theorem 2.25] we have the following bound for

the norm of the $\gamma_{t,s}$:

$$\|\gamma_{t,s}\| \leq e^{\int_s^t \|\mathcal{L}(\tau)\| d\tau} \leq e^{M_t(t-s)}.$$

Using again (3.2.1) we can rewrite the second factor as follows:

$$\begin{aligned} & \left(\text{id} + \frac{t}{n} \mathcal{L}\left(t \frac{j-1}{n}\right) \right) - \gamma_{t \frac{j}{n}, t \frac{j-1}{n}} = \frac{t}{n} \mathcal{L}\left(t \frac{j-1}{n}\right) - (\gamma_{t \frac{j}{n}, t \frac{j-1}{n}} - \text{id}) \\ &= \int_{t \frac{j-1}{n}}^{t \frac{j}{n}} \left(\mathcal{L}\left(t \frac{j-1}{n}\right) - \mathcal{L}(s) \gamma_{s, t \frac{j-1}{n}} \right) ds \\ &= \int_{t \frac{j-1}{n}}^{t \frac{j}{n}} \left[\left(\mathcal{L}\left(t \frac{j-1}{n}\right) - \mathcal{L}(s) \right) - \mathcal{L}(s) (\gamma_{s, t \frac{j-1}{n}} - \text{id}) \right] ds \\ &= \int_{t \frac{j-1}{n}}^{t \frac{j}{n}} \left(\mathcal{L}\left(t \frac{j-1}{n}\right) - \mathcal{L}(s) \right) ds - \int_{t \frac{j-1}{n}}^{t \frac{j}{n}} \mathcal{L}(s) \int_{t \frac{j-1}{n}}^s \mathcal{L}(\tau) \gamma_{\tau, t \frac{j-1}{n}} d\tau ds. \end{aligned}$$

Therefore, the second factor is bounded from above by

$$\begin{aligned} \left\| \left(\text{id} + \frac{t}{n} \mathcal{L}\left(t \frac{j-1}{n}\right) \right) - \gamma_{t \frac{j}{n}, t \frac{j-1}{n}} \right\| &\leq \frac{t}{n} \epsilon_n + M_t^2 \int_{t \frac{j-1}{n}}^{t \frac{j}{n}} \int_{t \frac{j-1}{n}}^s e^{(\tau - t \frac{j-1}{n}) M_t} d\tau ds \\ &\leq \frac{t}{n} \epsilon_n + M_t^2 e^{\frac{t}{n} M_t} \int_{t \frac{j-1}{n}}^{t \frac{j}{n}} \left(s - t \frac{j-1}{n} \right) ds \\ &= \frac{t}{n} \left(\epsilon_n + M_t^2 e^{\frac{t}{n} M_t} \frac{t}{2n} \right), \end{aligned}$$

where $\epsilon_n \rightarrow 0$ as $t/n \rightarrow 0$ due to the uniform continuity of $\mathcal{L}(t)$ on the interval $[0, t]$.

The third factor can be estimated in a similar way:

$$\begin{aligned} \|\gamma_{t \frac{j-1}{n}, 0}\| &= \prod_{k=j-1}^1 \|\gamma_{t \frac{k}{n}, t \frac{k-1}{n}}\| = \prod_{k=j-1}^1 \left\| 1 + \frac{t}{n} \mathcal{L}\left(s_k \left(\frac{t}{n}\right)\right) \right\| \\ &\leq \prod_{k=j-1}^1 \left(1 + \frac{t}{n} \|\mathcal{L}(s_k(\frac{t}{n}))\| \right) \\ &\leq \left(1 + \frac{t}{n} M_t \right)^n. \end{aligned}$$

Therefore, combining all these estimates we obtain

$$\begin{aligned} \|T_n(t) - \gamma_{t,0}\| &\leq n(1 + \frac{t}{n}M_t)^n \frac{t}{n} \left(\epsilon_n + M_t^2 e^{\frac{t}{n}M_t} \frac{t}{2n} \right) (1 + \frac{t}{n}M_t)^n \\ &\leq t e^{2tM_t} \left(\epsilon_n + M_t^2 e^{\frac{t}{n}M_t} \frac{t}{2n} \right). \end{aligned}$$

This bound vanishes as $n \rightarrow \infty$. □

To prove Theorem 4 we use the Euler-type approximation established in Lemma 1. We show that the action of $T_n(t)$ on a positive operator is bounded from below and this bound vanishes as n goes to ∞ .

PROOF OF THEOREM 4: First, we look at the each term in the Euler approximation $T_n(t)$ separately. For any t and s the complete dissipativity property (iii) of $\mathcal{L}(s)$, assumed in the statement of the theorem, implies

$$\begin{aligned} 0 &\leq (\text{id} + t\mathcal{L}(s))(A^*)(\text{id} + t\mathcal{L}(s))(A) = (A^* + t\mathcal{L}(s)(A^*))(A + t\mathcal{L}(s)(A)) \\ &= A^*A + tA^*\mathcal{L}(s)(A) + t\mathcal{L}(s)(A^*)A + t^2\mathcal{L}(s)(A^*)\mathcal{L}(s)(A) \\ &\leq A^*A + t\mathcal{L}(s)(A^*A) + t^2\mathcal{L}(s)(A^*)\mathcal{L}(s)(A). \end{aligned}$$

Since $(\mathcal{L}(s)(A))^*(\mathcal{L}(s)(A)) \leq \|\mathcal{L}(s)\|^2\|A\|$, one gets

$$0 \leq (\text{id} + t\mathcal{L}(s))(A^*A) + t^2\|\mathcal{L}(s)\|^2\|A\|^2 \tag{3.2.3}$$

$$\leq (\text{id} + t\mathcal{L}(s))(A^*A) + t^2M_s^2\|A\|^2. \tag{3.2.4}$$

Let us apply the above inequality to the operator B , where $B^*B := \|A\|^2 - A^*A$.

Note that $\|B^*B\| \leq \|A\|^2$, so $\|B\| \leq \|A\|$.

$$0 \leq (\text{id} + t\mathcal{L}(s))(\|A\|^2 - A^*A) + t^2 M_s^2 \|A\|^2 \quad (3.2.5)$$

$$= \|A\|^2 - (\text{id} + t\mathcal{L}(s))(A^*A) + t^2 M_s^2 \|A\|^2 \quad (3.2.6)$$

From the (3.2.3) and (3.2.5) we obtain

$$-t^2 M_s^2 \|A\|^2 \leq (\text{id} + t\mathcal{L}(s))(A^*A) \leq (1 + t^2 M_s^2) \|A\|^2 \quad (3.2.7)$$

and therefore:

$$-(1 + t^2 M_s^2) \|A\|^2 \leq (\text{id} + t\mathcal{L}(s))(A^*A) \leq (1 + t^2 M_s^2) \|A\|^2.$$

So we get

$$\|(\text{id} + t\mathcal{L}(s))(A^*A)\| \leq (1 + t^2 M_s^2) \|A\|^2. \quad (3.2.8)$$

Now, in order to bound the approximation $T_n(t)$ we first derive the following auxiliary estimate. For any fixed $n \geq 1$ we have:

$$\prod_{k=n}^1 (\text{id} + s\mathcal{L}(ks))(A^*A) \geq -s^2 \|A\|^2 M_{ns}^2 \left(1 + \frac{1}{n-1}\right)^{n-1} \sum_{k=0}^{n-1} D(s)^k, \quad (3.2.9)$$

where the value of s is chosen to be such that

$$D(s) := 1 + s^2 M_{ns}^2 < \left(1 + \frac{1}{n-1}\right)^{n-1} / \left(1 + \frac{1}{n-2}\right)^{n-2}, \quad (3.2.10)$$

with the convention that $\left(1 + \frac{1}{n-1}\right)^{n-1} = 1$, for $n = 1$.

We prove this claim by induction. The statement holds for $n = 1$ by (3.2.5). Now,

assume that (3.2.9) holds for $n - 1$. Then

$$\prod_{k=n-1}^1 (\text{id} + s\mathcal{L}(ks))(A^*A) + s^2\|A\|^2 M_{(n-1)s}^2 (1 + \frac{1}{n-2})^{n-2} \sum_{k=0}^{n-2} D(s)^k \geq 0$$

Since the left-hand side is a positive operator, we can write it as B^*B . Then,

$$\begin{aligned} \prod_{k=n}^1 (\text{id} + s\mathcal{L}(ks))(A^*A) &= (\text{id} + s\mathcal{L}(ns))(B^*B) \\ &\quad - s^2\|A\|^2 M_{(n-1)s}^2 (1 + \frac{1}{n-2})^{n-2} \sum_{k=0}^{n-2} D(s)^k \\ &\geq -s^2 M_{ns}^2 \|B^*B\| - s^2\|A\|^2 M_{ns}^2 (1 + \frac{1}{n-2})^{n-2} \sum_{k=0}^{n-2} D(s)^k. \end{aligned}$$

Here, we used (3.2.5) and the fact that M_t is monotone increasing. This gives the following upper bound for $\|B^*B\|$:

$$\begin{aligned} \|B^*B\| &\leq \prod_{k=n-1}^1 \|(\text{id} + s\mathcal{L}(ks))(A^*A)\| + s^2\|A\|^2 M_{(n-1)s}^2 (1 + \frac{1}{n-2})^{n-2} \sum_{k=0}^{n-2} D(s)^k \\ &\leq \prod_{k=n-1}^1 (1 + s^2 M_{ks}^2) \|A\|^2 + s^2\|A\|^2 M_{ns}^2 (1 + \frac{1}{n-2})^{n-2} \sum_{k=0}^{n-2} D(s)^k \\ &\leq \prod_{k=n-1}^1 (1 + s^2 M_{ns}^2) \|A\|^2 + s^2\|A\|^2 M_{ns}^2 (1 + \frac{1}{n-2})^{n-2} \sum_{k=0}^{n-2} D(s)^k \\ &= \|A\|^2 D(s)^{n-1} + s^2\|A\|^2 M_{ns}^2 (1 + \frac{1}{n-2})^{n-2} \sum_{k=0}^{n-2} D(s)^k \end{aligned}$$

Therefore we obtain

$$\begin{aligned}
& \prod_{k=n-1}^1 (1 + s\mathcal{L}(ks))(A^*A) \\
& \geq -s^2 M_{ns}^2 \|A\|^2 D(s)^{n-1} - s^2 M_{ns}^2 (s^2 M_{ns}^2 + 1) \left(1 + \frac{1}{n-2}\right)^{n-2} \|A\|^2 \sum_{k=0}^{n-2} D(s)^k \\
& \geq -s^2 M_{ns}^2 \|A\|^2 \left(1 + \frac{1}{n-1}\right)^{n-1} D(s)^{n-1} - s^2 M_{ns}^2 \left(1 + \frac{1}{n-1}\right)^{n-1} \|A\|^2 \sum_{k=0}^{n-2} D(s)^k \\
& \geq -s^2 M_{ns}^2 \left(1 + \frac{1}{n-1}\right)^{n-1} \|A\|^2 \sum_{k=0}^{n-1} D(s)^k,
\end{aligned}$$

where to pass to the second inequality we use our assumption on s (3.2.10). This completes the proof of the bound (3.2.9).

To finish the proof of the theorem we use Lemma 1 to approximate the propagator and put $s = \frac{t}{n}$ in the bound (3.2.9), which yields

$$\prod_{k=n}^1 \left(1 + \frac{t}{n} \mathcal{L}\left(\frac{kt}{n}\right)\right)(A^*A) \geq -\frac{t^2}{n^2} \|A\|^2 M_t^2 \left(1 + \frac{1}{n-1}\right)^{n-1} \sum_{k=0}^{n-1} D\left(\frac{t}{n}\right)^k. \quad (3.2.11)$$

Since $D\left(\frac{t}{n}\right)^n = \left(1 + \frac{t^2}{n^2} M_t^2\right)^n \rightarrow 1$ as $n \rightarrow \infty$, we get the estimate $D\left(\frac{t}{n}\right)^k \leq 2$ for $1 \leq k \leq n$. The right hand side of (3.2.11) is bounded from below by $-\frac{t^2}{n^2} \|A\|^2 e M_t^2 2n$, which vanishes in the limit $n \rightarrow \infty$.

To show the complete positivity of $\gamma_{t,0}$ note that any generator $\mathcal{L}_\Lambda(t)$ satisfying the assumptions of the theorem can be considered as the generator for a dynamics on $\mathcal{A} \otimes \mathcal{B}(\mathbb{C}^n)$, for any $n \geq 1$, which satisfies the same properties, and which generates $\gamma_{t,s} \otimes \text{id}$ acting on $\mathcal{A} \otimes \mathcal{B}(\mathbb{C}^n)$. By the arguments given above, these maps are positive for all n . Hence, the $\gamma_{t,s}$ are completely positive. \square

3.3 Lieb-Robinson bounds for a class of irreversible dynamics

For $X \subset \Lambda$, let \mathcal{B}_X denote the subspace of $\mathcal{B}(\mathcal{A}_X)$ consisting of all completely bounded linear maps that vanish on $\mathbb{1}$.

It is important for us that all operators of the form

$$\mathcal{K}_X(B) := i[A, B] + \sum_{a=1}^N (L_a^* B L_a - \frac{1}{2} \{L_a^* L_a, B\}),$$

where $A, L_a \in \mathcal{A}_X$, belong to \mathcal{B}_X , with

$$\|\mathcal{K}_X\|_{\text{cb}} \leq 2\|A\| + 2 \sum_{a=1}^N \|L_a\|^2.$$

In particular, operators of the form $[A, \cdot]$ appearing in the standard Lieb-Robinson bound (2.2.7) are a special case of this general form.

We can regard \mathcal{K}_X as a linear transformation on \mathcal{A}_Z , for all Z such that $X \subset Z$, by tensoring it with $\text{id}_{\mathcal{A}_{Z \setminus X}}$, and all these maps will be bounded with norm less than $\|\mathcal{K}_X\|_{\text{cb}}$.

Theorem 5. *Suppose Assumption 1 holds. Then the maps $\gamma_{t,s}^\Lambda$ satisfy the following bound. For $X, Y \subset \Lambda$, and any operators $\mathcal{K} \in \mathcal{B}_X$ and $B \in \mathcal{A}_Y$ we have that*

$$\|\mathcal{K} \gamma_{t,s}^\Lambda(B)\| \leq \frac{\|\mathcal{K}\|_{\text{cb}} \|B\|}{C_\mu} e^{\|\Psi\|_{t,\mu} C_\mu |t-s|} \sum_{x \in X \subset \Lambda} \sum_{y \in Y \subset \Lambda} F(d(x, y)).$$

The proof of the Lieb-Robinson bounds for $\gamma_{t,s}^\Lambda$ is based on a generalization of the strategy [15] for reversible dynamics, and on [20] for irreversible dynamics with time-independent generators. This allows us to cover the case of irreversible dynamics

with time-dependent generators.

PROOF OF THEOREM 5: Consider the function $f : [s, \infty) \rightarrow \mathcal{A}$ defined by

$$f(t) = \mathcal{K} \gamma_{t,s}^\Lambda(B),$$

where $\mathcal{K} \in \mathcal{B}_X$ and $B \in \mathcal{A}_Y$, as in the statement of the theorem. For $X \subset \Lambda$, let $X^c = \Lambda \setminus X$ and define \mathcal{L}_{X^c} and $\bar{\mathcal{L}}_X$ by

$$\begin{aligned} \mathcal{L}_{X^c}(t) &= \sum_{Z, Z \cap X = \emptyset} \mathcal{L}_Z(t) \\ \bar{\mathcal{L}}_X(t) &= \mathcal{L}_X(t) - \mathcal{L}_{X^c}(t). \end{aligned}$$

Clearly, $[\mathcal{K}, \mathcal{L}_{X^c}(t)] = 0$. Using this property, we easily derive the following expression for the derivative of f :

$$\begin{aligned} f'(t) &= \mathcal{K} \mathcal{L}(t) \gamma_{t,s}^\Lambda(B) \\ &= \mathcal{L}_{X^c}(t) \mathcal{K} \gamma_{t,s}^\Lambda(B) + \mathcal{K} \bar{\mathcal{L}}_X(t) \gamma_{t,s}^\Lambda(B) \\ &= \mathcal{L}_{X^c}(t) f(t) + \mathcal{K} \bar{\mathcal{L}}_X(t) \gamma_{t,s}^\Lambda(B), \end{aligned}$$

Let $\gamma_{t,s}^{X^c}$ be the cocycle generated by $\mathcal{L}_{X^c}(t)$. Then, using the expression for $f'(t)$ we find

$$f(t) = \gamma_{t,s}^{X^c} f(s) + \int_s^t \gamma_{t,r}^{X^c} \mathcal{K} \bar{\mathcal{L}}_X(r) \gamma_{r,s}^\Lambda(B) dr.$$

Since $\gamma_{t,s}^{X^c}$ is norm-contracting and $\|\mathcal{K}\|_{\text{cb}}$ is an upper bound for the $\|\mathcal{K}\|$ regarded as an operator on \mathcal{A}_Λ , for all Λ , we obtain

$$\|f(t)\| \leq \|f(s)\| + \|\mathcal{K}\|_{\text{cb}} \int_s^t \|\bar{\mathcal{L}}_X(r) \gamma_{r,s}^\Lambda(B)\| dr. \quad (3.3.1)$$

Let us define the quantity

$$C_B(X, t) := \sup_{\mathcal{T} \in \mathcal{B}_X} \frac{\|\mathcal{T}\gamma_{t,s}^\Lambda(B)\|}{\|\mathcal{T}\|_{\text{cb}}}.$$

Note that we use the norm $\|\mathcal{T}\|_{\text{cb}}$, because, as mentioned before and in contrast to the usual operator norm, it is independent of Λ . Then, we have the following obvious estimate:

$$C_B(X, s) \leq \|B\| \delta_Y(X),$$

where $\delta_Y(X) = 0$ if $X \cap Y = \emptyset$ and $\delta_Y(X) = 1$ otherwise. From the definition of the space \mathcal{B}_X we get that $\mathcal{T}(B) = 0$, when $\mathcal{T} \in \mathcal{B}_X$, since B has a support in Y and $Y \cap X = \emptyset$.

Therefore (3.3.1) implies that

$$C_B(X, t) \leq C_B(X, s) + \sum_{Z \cap X \neq \emptyset} \int_s^t \|\mathcal{L}_Z(r)\| C_B(Z, r) dr.$$

Iterating this inequality we find the estimate:

$$C_B(X, t) \leq \|B\| \sum_{n=0}^{\infty} \frac{(t-s)^n}{n!} a_n,$$

where:

$$a_n \leq \|\Psi\|_{t,\mu}^n C_\mu^{n-1} \sum_{x \in X} \sum_{y \in Y} F(d(x, y)),$$

for $n \geq 1$ and $a_0 = 1$, (recall that C_μ is a constant, that appears in a definition of F_μ). The following bound immediately follows from this estimate:

$$\|\mathcal{K}\gamma_{t,s}^\Lambda(B)\| \leq \frac{\|\mathcal{K}\|_{\text{cb}} \|B\|}{C_\mu} e^{\|\Psi\|_{t,\mu} C_\mu (t-s)} \sum_{x \in X \cap \Lambda} \sum_{y \in Y \cap \Lambda} F(d(x, y)).$$

If we take an exponentially decaying function F , defined in (2.2.5) as $F_\mu(d) = e^{-\mu d} F(d)$ we can rewrite the above bound as follows

$$\|\mathcal{K}\gamma_{t,s}^\Lambda(B)\| \leq \frac{\|\mathcal{K}\|_{\text{cb}}\|B\|}{C_\mu} \|F\| \min(|X|, |Y|) e^{-\mu(d(X,Y) - \frac{\|\Psi\|_{t,\mu} C_\mu}{\mu}(t-s))}. \quad (3.3.2)$$

So the Lieb-Robinson velocity of the propagation for every $t \in \mathbf{R}$ is

$$v_{t,\mu} := \frac{\|\Psi\|_{t,\mu} C_\mu}{\mu}.$$

□

Note that the Lieb-Robinson bound (3.3.2) depends only on the smallest of the supports of the two observables. Therefore one can get a non-trivial bound even when one of the observables has finite support but the support of the other is of infinite size (e.g., say half the system).

The bound in the Lieb-Robinson bound is *uniform* in Λ . This is important for the proof of existence of the thermodynamic limit of the dynamics, which is one of the main applications of Lieb-Robinson bounds, which will be presented in the next section.

3.4 Existence of the thermodynamic limit

The setup for the analysis of the thermodynamic limit can be formulated as follows. Let Γ be an infinite set such as, e.g., the hypercubic lattice \mathbb{Z}^ν . We prove the existence of the thermodynamic limit for an increasing and exhausting sequence of finite subsets $\Lambda_n \subset \Gamma$, $n \geq 1$, by showing that for each $A \in \mathcal{A}_X$, $(\gamma_{t,s}^{\Lambda_n}(A))_{n \geq 1}$ is a Cauchy sequence in the norm of \mathcal{A}_Γ . To this end we have to suppose that Assumption 1 (2) holds

uniformly for all Λ_n , i.e., we can replace Λ in (3.1.3) by Γ .

Theorem 6. *Suppose that Assumption 1 holds and, in addition, that (3.1.3) holds for $\Lambda = \Gamma$. Then, there exists a strongly continuous cocycle of unit-preserving completely positive maps $\gamma_{t,s}^\Gamma$ on \mathcal{A}_Γ such that for all $0 \leq s \leq t$, and any increasing exhausting sequence of finite subsets $\Lambda_n \subset \Gamma$, we have*

$$\lim_{n \rightarrow \infty} \|\gamma_{t,s}^{\Lambda_n}(A) - \gamma_{t,s}^\Gamma(A)\| = 0, \quad (3.4.1)$$

for all $A \in \mathcal{A}_\Gamma$.

The proof of existence of the thermodynamic limit mimics the method given in the paper [15].

PROOF OF THEOREM 6: Denote $\mathcal{L}_n = \mathcal{L}_{\Lambda_n}$ and $\gamma_{t,s}^{\Lambda_n} = \gamma_{t,s}^{(n)}$. Let $n > m$, then $\Lambda_m \subset \Lambda_n$ since we have the exhausting sequence of subsets in Γ . We will prove that for every observable $A \in \mathcal{A}_X$ the sequence $(\gamma_{t,s}^n(A))_{n \geq 1}$ is a Cauchy sequence. In order to do that for any local observable $A \in \mathcal{A}_X$ we consider the function

$$f(t) := \gamma_{t,s}^{(n)}(A) - \gamma_{t,s}^{(m)}(A) .$$

Calculating the derivative, we obtain

$$\begin{aligned} f'(t) &= \mathcal{L}_n \gamma_{t,s}^{(n)}(A) - \mathcal{L}_m \gamma_{t,s}^{(m)}(A) \\ &= \mathcal{L}_n(t)(\gamma_{t,s}^{(n)}(A) - \gamma_{t,s}^{(m)}(A)) + (\mathcal{L}_n(t) - \mathcal{L}_m(t))\gamma_{t,s}^{(m)}(A) \\ &= \mathcal{L}_n(t)f(t) + (\mathcal{L}_n(t) - \mathcal{L}_m(t))\gamma_{t,s}^{(m)}(A). \end{aligned}$$

The solution to this differential equation is

$$f(t) = \int_s^t \gamma_{t,r}^{(n)}([\mathcal{L}_n(r) - \mathcal{L}_m(r)]\gamma_{r,s}^{(m)}(A))dr.$$

Since $\gamma_{t,r}^{(n)}$ is norm-contracting, from this formula we get the estimate:

$$\begin{aligned} \|f(t)\| &\leq \int_s^t \|(\mathcal{L}_n(r) - \mathcal{L}_m(r))\gamma_{r,s}^{(m)}(A)\|dr \\ &\leq \sum_{z \in \Lambda_n \setminus \Lambda_m} \sum_{Z \ni z} \int_s^t \|\Psi_Z(r)(\gamma_{r,s}^{(m)}(A))\|dr. \end{aligned}$$

Using the Lieb-Robinson bound and the exponential decay condition (3.1.3), which we assumed holds uniformly in Λ , we find that

$$\begin{aligned} \|f(t)\| &\leq \frac{\|A\|}{C_\mu} \int_s^t e^{\mu v_{r,\mu}(r-s)} \sum_{z \in \Lambda_n \setminus \Lambda_m} \sum_{Z \ni z} \|\Psi_Z(r)\|_{\text{cb}} \sum_{x \in X} \sum_{y \in Z} F(d(x, y))dr \\ &\leq \frac{\|A\|}{C_\mu} \int_s^t e^{\mu v_{r,\mu}(r-s)} \sum_{z \in \Lambda_n \setminus \Lambda_m} \sum_{x \in X} \sum_{y \in \Gamma} \sum_{Z \ni z, y} \|\Psi_Z(r)\|_{\text{cb}} F(d(x, y))dr \\ &\leq \frac{\|A\|}{C_\mu} \|\Psi\|_{t,\mu} \int_s^t e^{\mu v_{r,\mu}(r-s)} dr \sum_{z \in \Lambda_n \setminus \Lambda_m} \sum_{x \in X} \sum_{y \in \Gamma} F(d(x, y))F(d(y, z)) \\ &\leq \|A\| \|\Psi\|_{t,\mu} \int_s^t e^{\mu v_{r,\mu}(r-s)} dr \sum_{z \in \Lambda_n \setminus \Lambda_m} \sum_{x \in X} F(d(x, z)) \\ &\leq \|A\| \|\Psi\|_{t,\mu} \int_s^t e^{\mu v_{r,\mu}(r-s)} dr |X| \sup_{x \in X} \sum_{z \in \Lambda_n \setminus \Lambda_m} F(d(x, z)). \end{aligned}$$

Since F is uniformly integrable (2.2.3) the tail sum above goes to zero as $n, m \rightarrow \infty$.

Thus

$$\|(\gamma_{t,s}^{(n)} - \gamma_{t,s}^{(m)})(A)\| \rightarrow 0, \text{ as } n, m \rightarrow \infty.$$

Therefore the sequence $\{\gamma_{t,s}^{(n)}(A)\}_{n=0}^\infty$ is Cauchy and hence convergent. Denote the limit, and its extension to \mathcal{A}_Γ , as $\gamma_{t,s}^\Gamma$.

To show that $\gamma_{t,s}^\Gamma$ is strongly continuous we notice that for $0 \leq s \leq t, r \leq T$, and any $A \in \mathcal{A}_\Gamma^{\text{loc}}$, we have

$$\|\gamma_{t,s}^\Gamma(A) - \gamma_{r,s}^\Gamma(A)\| \leq \|\gamma_{t,s}^\Gamma(A) - \gamma_{t,s}^{(n)}(A)\| + \|\gamma_{t,s}^{(n)}(A) - \gamma_{r,s}^{(n)}(A)\| + \|\gamma_{r,s}^{(n)}(A) - \gamma_{r,s}^\Gamma(A)\|,$$

for any $n \in \mathbb{N}$ such that $A \in \mathcal{A}_{\Lambda_n}$.

The strong continuity then follows from the strong convergence of $\gamma_{t,s}^{(n)}$ to $\gamma_{t,s}^\Gamma$, uniformly in $s \leq t \in [0, T]$, and the strong continuity of $\gamma_{t,s}^{(n)}$ in t . The continuity of the extension of $\gamma_{t,s}^\Gamma$ to all of $A \in \mathcal{A}_\Gamma$ follows by the standard density argument. The argument for continuity in the second variable, s , is similar. \square

Chapter 4

Non-equilibrium state of a leaking photon cavity pumped by a random atomic beam

4.1 Set up

4.1.1 Description of the model

Our model consists of a beam of two-level atoms that passe one-by-one a photon cavity \mathcal{C} . Atoms in a beam are randomly excited. During the passage time τ the corresponding single atom is able to interact with the cavity. For simplicity we consider a, so called, *tuned* case, when the cavity size is equal to the interatomic distance, so there is always one atom in the cavity.

The cavity is a one-mode resonator described by quantum harmonic oscillator with the Hamiltonian $H_C = \epsilon b^* b \otimes \mathbb{1}$ in the Hilbert space \mathcal{H}_C , where b^* and b stand for boson (photon) creation and annihilation operators with canonical commutation

relations (CCR): $[b, b^*] = 1$, $[b, b] = [b^*, b^*] = 0$.

The beam of two-level, $\{E, 0\}$, atoms is described as a chain $H_{\mathcal{A}} = \sum_{n \geq 1} H_{\mathcal{A}_n}$ of individual atoms with Hamiltonian $H_{\mathcal{A}_n} = \mathbb{1} \otimes E \eta_n$ in the Hilbert space $\mathcal{H}_{\mathcal{A}} = \otimes_{n \geq 1} \mathcal{H}_{\mathcal{A}_n}$. Here for any $n \geq 1$, $\mathcal{H}_{\mathcal{A}_n} = \mathbb{C}^2$ and the individual atomic operator $\eta_n := (\sigma^z + \mathbb{1})/2$, where σ^z is the third Pauli matrix. The eigenvectors ψ_n^\pm : $\eta_n \psi_n^+ = \psi_n^+$ and $\eta_n \psi_n^- = 0$, are interpreted as the excited and the ground states of the atom, respectively.

The initial state of the system is the product state of the cavity and the states of each individual atom:

$$\rho_S := \rho_C \otimes \bigotimes_{k \geq 1} \rho_k . \quad (4.1.1)$$

Here ρ_C is the initial state of the cavity, which we assume to be normal, i.e., given by a density matrix $\rho_C \in \mathfrak{C}_1(\mathcal{H}_C)$, where \mathfrak{C}_1 denotes the space of the trace-class operators on \mathcal{H}_C , and $\rho_{\mathcal{A}} := \bigotimes_{k \geq 1} \rho_k$ is the state of the atomic beam. The states $\{\rho_k\}_{k \geq 1}$ on the algebras $\{\mathfrak{C}_1(\mathcal{H}_{\mathcal{A}_k})\}_{k \geq 1}$ are diagonal and identical (homogeneous beam), hence of the form

$$\rho_k = \begin{pmatrix} p & 0 \\ 0 & 1 - p \end{pmatrix} .$$

Here, the parameter $p := \text{Tr}_{\mathcal{H}_{\mathcal{A}_k}}(\eta_k \rho_k)$ denotes the probability with which each atom is in its excited state ψ_k^+ .

In our model the repeated cavity-atom interaction is time-dependend and piecewise constant, that has the form:

$$K_n(t) = \chi_{[(n-1)\tau, n\tau)}(t) \lambda (b^* + b) \otimes \eta_n . \quad (4.1.2)$$

Here $\chi_I(x)$ is characteristic function of the set I .

The Hamiltonian of the entire system acts on the space $\mathcal{H}_S := \mathcal{H}_C \otimes \mathcal{H}_A$, and is given by the sum of the Hamiltonian of the cavity, of the atoms, and the interaction between them

$$\begin{aligned} H(t) &= H_C + \sum_{n \geq 1} (H_{A_n} + K_n(t)) \\ &= \epsilon b^* b \otimes \mathbb{1} + \sum_{n \geq 1} \mathbb{1} \otimes E \eta_n + \sum_{n \geq 1} \chi_{[(n-1)\tau, n\tau)}(t) (\lambda (b^* + b) \otimes \eta_n) . \end{aligned} \quad (4.1.3)$$

Notice that for the time $t \in [(n-1)\tau, n\tau)$, only the n -th atom interacts with the cavity and the Hamiltonian is time-independent.

4.1.2 Hamiltonian dynamics of perfect cavity

Let $t \in [(n-1)\tau, n\tau)$. Then the Hamiltonian (4.1.3) for the n -th atom in the cavity takes the form

$$H_n := \epsilon b^* b \otimes \mathbb{1} + \mathbb{1} \otimes E \eta_n + \lambda (b^* + b) \otimes \eta_n . \quad (4.1.4)$$

Even though the atomic beam is infinite, there is only a finite number of atoms that have interacted with the cavity at any finite time t . Assuming that the initial state of the system is normal, our system can be described by normal states $\omega_S(\cdot) := \text{Tr}(\cdot \rho_S)$, which are defined by trace-class density matrices $\rho_S \in \mathfrak{C}_1(\mathcal{H}_C \otimes \mathcal{H}_A)$.

Then the partial traces over \mathcal{H}_A and over \mathcal{H}_C :

$$\omega_C(\cdot) := \text{Tr}_{\mathcal{H}_A}(\cdot \rho_S) \quad \text{and} \quad \omega_A(\cdot) := \text{Tr}_{\mathcal{H}_C}(\cdot \rho_S) , \quad (4.1.5)$$

define respectively the cavity and the beam states.

We often refer to density matrices as states, if this wording will not produce any confusion.

We suppose that initially our system is in a product state:

$$\omega_S(\cdot) := (\omega_C \otimes \omega_A)(\cdot) = \text{Tr}(\cdot \rho_C \otimes \rho_A) , \quad \rho_S = \rho_C \otimes \rho_A , \quad (4.1.6)$$

where $\rho_C \in \mathfrak{C}_1(\mathcal{H}_C)$ and $\rho_A \in \mathfrak{C}_1(\mathcal{H}_A)$.

For any state ρ_S on the algebra of observables $\mathfrak{A}(\mathcal{H}_C \otimes \mathcal{H}_A)$ the Hamiltonian dynamics of the system is defined by (4.1.3), or by the quantum time-dependent Liouvillian generator:

$$L(t)(\rho_S) := -i [H(t), \rho_S] . \quad (4.1.7)$$

Then the state $\rho_S(t)$ of the total system at the time t is a solution of the Cauchy problem corresponding to Liouville differential equation

$$\frac{d}{dt}\rho_S(t) = L(t)(\rho_S(t)) , \quad \rho_S(t=0) := \rho_C \otimes \rho_A . \quad (4.1.8)$$

We denote by $\omega_S^t(\cdot) := \text{Tr}(\cdot \rho_S(t))$ the system time evolution due to (4.1.8) for the initial product state $\omega_S(\cdot)$ (4.1.6).

Notice that in general the Hamiltonian evolution (4.1.8) with time-dependent generator is a family of automorphisms $\{\Gamma_{t,s}\}_{0 \leq s \leq t}$ with the *cocycle property*:

$$\rho_S(t) = \Gamma_{t,s} \rho_S(s) . \quad (4.1.9)$$

For our model with tuned repeated interactions the solution of (4.1.8) and the form of the evolution operator (4.1.9) are considerably simplified. Indeed, the generator of the dynamics of our system (4.1.3) is time-independent for each interval $[(n-1)\tau, n\tau)$.

Therefore, by virtue of (4.1.4) and (4.1.7), the Liouvillian generators

$$L_n := L(t) , \ t \in [(n-1)\tau, n\tau) , \ n \geq 1 , \quad (4.1.10)$$

are time-independent and commuting. Note that any moment $t \geq 0$ has the representation

$$t := n(t)\tau + \nu(t) , \ n(t) = [t/\tau] \text{ and } \nu(t) \in [0, \tau) , \quad (4.1.11)$$

where $[x]$ denotes the integer part of the number $x \in \mathbb{R}$. Then solution of (4.1.8) for $t \in [(n-1)\tau, n\tau)$ gets the form:

$$\rho_S(t) = \Gamma_{t,s=0}(\rho_C \otimes \rho_A) = e^{\nu L_n} e^{\tau L_{n-1}} \dots e^{\tau L_2} e^{\tau L_1}(\rho_C \otimes \rho_A) . \quad (4.1.12)$$

In the next section 4.2 we exploit a specific structure of the Hamiltonian dynamics (4.1.12) and a special form of interaction (4.1.2) to work out an effective evolution of the perfect cavity H_C . Our results concern first of all the evolution of the photon number expectation in the time-dependent cavity state

$$\rho_C(t) := \text{Tr}_{\mathcal{H}_A} \rho_S(t) . \quad (4.1.13)$$

Note that by (4.1.3),(4.1.4) one gets that in our model the atomic states ρ_k do not evolve:

$$[\mathbb{1} \otimes \rho_k, H(t)] = 0 , \ k \geq 1 . \quad (4.1.14)$$

The form of the initial state (4.1.1) together with (4.1.12),(4.1.13) imply a discrete

time evolution for the cavity state given by the following recursive formula:

$$\begin{aligned}
\rho_C^{(n)} &:= \rho_C(t = n\tau) = \text{Tr}_{\mathcal{H}_A} \rho_S(n\tau) = \text{Tr}_{\mathcal{H}_A} [e^{\tau L_n} \dots e^{\tau L_2} e^{\tau L_1} (\rho_C \otimes \bigotimes_{k=1}^n \rho_k)] \quad (4.1.15) \\
&= \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}} [e^{\tau L_n} \{ \text{Tr}_{\mathcal{H}_{\mathcal{A}_{n-1}}} \dots \text{Tr}_{\mathcal{H}_{\mathcal{A}_1}} e^{\tau L_{n-1}} \dots e^{\tau L_2} e^{\tau L_1} (\rho_C \otimes \bigotimes_{k=1}^{n-1} \rho_k) \} \otimes \rho_n] \\
&= \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}} [e^{\tau L_n} (\rho_C^{(n-1)} \otimes \rho_n)] .
\end{aligned}$$

For any density matrix $\rho \in \mathfrak{C}_1(\mathcal{H}_C)$ corresponding to normal state on the operator algebra $\mathfrak{A}(\mathcal{H}_C)$ we define the mapping $\mathcal{L} : \rho \mapsto \mathcal{L}(\rho)$, by

$$\mathcal{L}(\rho) := \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}} (e^{\tau L_n} (\rho \otimes \rho_n)) = \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}} [e^{-i\tau H_n} (\rho \otimes \rho_n) e^{i\tau H_n}] . \quad (4.1.16)$$

Here the last equality is due to (4.1.7) and (4.1.10).

Note that the mapping (4.1.16) does not depend on $n \geq 1$, since the atomic states $\{\rho_n\}_{n \geq 1}$ are homogeneous. Then the cavity state at $t = n\tau$ is defined by the n -th power of \mathcal{L} :

$$\rho_C^{(n)} = \mathcal{L}(\rho_C^{(n-1)}) = \mathcal{L}^n(\rho_C) . \quad (4.1.17)$$

Therefore, by (4.1.12), (4.1.15) and (4.1.17) one obtains that for any time $t = n\tau + \nu$, where $\nu \in [0, \tau)$, the cavity state is

$$\rho_C(t) = \text{Tr}_{\mathcal{H}_{\mathcal{A}_{n+1}}} [e^{\nu L_{n+1}} (\mathcal{L}^n(\rho_C) \otimes \rho_{n+1})] . \quad (4.1.18)$$

We are interested in the expectation value $N(t)$ of the photon-number operator $\widehat{N} := b^*b$ in the cavity at the time t :

$$N(t) := \omega_C^t(\widehat{N}) = \text{Tr}_{\mathcal{H}_C} (b^*b \rho_C(t)) . \quad (4.1.19)$$

For $t = n\tau$ the state of the cavity can be expressed using (4.1.17), which gives

$$N(n\tau) = \text{Tr}_{\mathcal{H}_c}(b^*b \mathcal{L}^n(\rho_c)) . \quad (4.1.20)$$

4.1.3 Quantum dynamics of leaking cavity

For a more general situation we take into account a possible leakage of the cavity, where photons may leave the cavity at some non-zero rate $\sigma > 0$. We consider this case in the framework of Kossakowski-Lindblad extension of the Hamiltonian Dynamics to irreversible Quantum Dynamics with time-dependent generator

$$L_\sigma(t)(\rho_S) = -i[H(t), \rho_S] + \sigma b(\rho_S)b^* - \frac{\sigma}{2}\{b^*b, \rho_S\} , \quad (4.1.21)$$

for any $\rho_S \in \mathfrak{C}_1(\mathcal{H}_c \otimes \mathcal{H}_A)$.

Similar to (4.1.8) the evolution of the state is defined by a solution of the non-autonomous Cauchy problem corresponding to time-dependent generator (4.1.21)

$$\frac{d}{dt}\rho_S(t) = L_\sigma(t)(\rho_S(t)) , \quad \rho_S(t=0) := \rho_c \otimes \rho_A . \quad (4.1.22)$$

In general for a time-dependent generator $L_\sigma(t)$ the proof of existence of this solution is a non-trivial problem, as in chapter 3. In the present case of the tuned repeated interactions, when the Hamiltonian is time-independent for each interval $[(n-1)\tau, n\tau)$, the generator (4.1.21) gets the form:

$$L_{\sigma,n}(\rho_S) := -i[H_n, \rho_S] + \sigma b(\rho_S)b^* - \frac{\sigma}{2}\{b^*b, \rho_S\} , \quad \rho_S = \rho_c \otimes \rho_A . \quad (4.1.23)$$

With restriction to the tuned case and with the help of (4.1.23) the solution of

(4.1.22) for $t \in [(n-1)\tau, n\tau)$ becomes of the form:

$$\rho_S(t) = U_{t,s=0}^\sigma(\rho_C \otimes \rho_A) = e^{\nu L_{\sigma,n}} e^{\tau L_{\sigma,n-1}} \dots e^{\tau L_{\sigma,2}} e^{\tau L_{\sigma,1}}(\rho_C \otimes \rho_A) . \quad (4.1.24)$$

Here $\{U_{t,s}^\sigma\}_{0 \leq s \leq t}$ is a family of unit preserving completely positive maps with a cocycle property (4.1.9).

The discrete evolution operator (4.1.16) has to be modified for $\sigma > 0$ as follows:

Definition 1. *For any state ρ on $\mathfrak{A}(\mathcal{H}_C)$ we define the mapping*

$$\mathcal{L}_\sigma(\rho) := \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}}(e^{\tau L_{\sigma,n}}(\rho \otimes \rho_n)) . \quad (4.1.25)$$

Let $\rho_C := \rho_C(t=0)$ denote the initial state of the cavity. Then similar to (4.1.18) we obtain for $\rho_C(t)$ at the moment $t = n\tau + \nu$, where $\nu \in [0, \tau)$,

$$\rho_C(t) = \text{Tr}_{\mathcal{H}_{\mathcal{A}_{n+1}}}[e^{\nu L_{\sigma,n+1}}(\mathcal{L}_\sigma^n(\rho_C) \otimes \rho_{n+1})] . \quad (4.1.26)$$

Now we define the functional $\omega_C^t(\cdot) := \text{Tr}_{\mathcal{H}_C}(\cdot \rho_C(t))$. We study the infinite-time limit for the cavity state $\omega_C(\cdot) := \lim_{t \rightarrow \infty} \omega_C^t(\cdot)$. Here the limit means trace-norm convergence of the sequence (4.1.26) to density matrix $\rho_{C,\sigma}$:

$$\lim_{t \rightarrow \infty} \|\rho_{C,\sigma} - \rho_{C,\sigma}(t)\|_1 = 0, \quad (4.1.27)$$

where the norm $\|\cdot\|_1$ denotes the trace-norm on the space of the trace-class operators $\mathfrak{C}_1(\mathcal{H}_C)$.

We consider the functional

$$\omega_C(W(\alpha)) = \lim_{t \rightarrow \infty} \omega_C^t(W(\alpha)) , \quad (4.1.28)$$

generated by the Weyl operator on \mathcal{H}_C :

$$W(\alpha) = e^{\alpha b - \bar{\alpha} b^*}, \quad \alpha \in \mathbb{C}.$$

See Section 2.1.2 for more on Weyl operators. Notice that convergence (4.1.28) for the family of the Weyl operators guarantees the weak-* limit [2] of the states $\omega_C^t(\cdot)$ when $t \rightarrow \infty$.

4.2 Number of photons in perfect cavity

First we consider the case of the perfect cavity, i.e. $\sigma = 0$. Then for the discrete evolution operator one gets $\mathcal{L}_{\sigma=0} = \mathcal{L}$, see (4.1.16) and (4.1.25).

Our first result concerns the expectation of the photon-number operator $\hat{N} = b^*b$ in the cavity (4.1.19). For $t = n\tau$ this expectation value takes the form (4.1.20).

In the theorem below we suppose that the initial cavity state $\omega_C(\cdot)$ is also gauge invariant, i.e. $e^{i\alpha\hat{N}}\rho_C e^{-i\alpha\hat{N}} = \rho_C$.

Theorem 7. *Let ρ_C be a gauge invariant state. Then for $t = n\tau$ the expectation value (4.1.20) of the photon number in the cavity is*

$$N(t) = N(0) + np(1-p) \frac{2\lambda^2}{\epsilon^2} (1 - \cos \epsilon\tau) + p^2 \frac{2\lambda^2}{\epsilon^2} (1 - \cos n\epsilon\tau). \quad (4.2.1)$$

If for the initial state ρ_C one takes in the theorem the Gibbs state for photons at the inverse temperature β :

$$\rho_C^\beta = e^{-\beta\epsilon b^*b} / \text{Tr}_{\mathcal{H}_C} e^{-\beta\epsilon b^*b}, \quad (4.2.2)$$

then the number of photons (4.2.1) is

$$N(t) = \frac{1}{e^{\beta\epsilon} - 1} + n p(1-p) \frac{2\lambda^2}{\epsilon^2} (1 - \cos \epsilon\tau) + p^2 \frac{2\lambda^2}{\epsilon^2} (1 - \cos n\epsilon\tau) . \quad (4.2.3)$$

To prepare the proof of Theorem 7 we calculate first explicit expressions for $\mathcal{L}(\rho)$ acting on the space $\rho \in \mathfrak{C}(\mathcal{H}_C)$ of the cavity states and for the n -th power of the dual operator \mathcal{L}^* that we apply to the number operator b^*b , see (4.2.10) and Remark 1.

We split these calculations into two lemmas. The proof of the Theorem 7 is presented at the end of this section.

The Hamiltonian (4.1.4) can be written in the following form

$$\begin{aligned} H_n &= \epsilon \left(b^* + \frac{\lambda}{\epsilon} \eta_n \right) \left(b + \frac{\lambda}{\epsilon} \eta_n \right) + E \eta_n - \frac{\lambda^2}{\epsilon} \eta_n \\ &= \epsilon \hat{b}^* \hat{b} + (E - \lambda^2/\epsilon) \eta_n . \end{aligned}$$

Here $\hat{b} := b + \lambda \eta_n / \epsilon$ and $\hat{b}^* := b^* + \lambda \eta_n / \epsilon$ are new boson operators since by the CAR and CCR properties of η_n^* and respectively b, b^* one gets: $[\hat{b}, \hat{b}^*] = [b, b^*] = 1$ and $[\hat{b}, \hat{b}] = [\hat{b}^*, \hat{b}^*] = 0$ for *any* index n .

This is in fact a unitary shift transformation generated by

$$V_n := \frac{\lambda}{i\epsilon} (b^* - b) \otimes \eta_n . \quad (4.2.4)$$

Then the transformed Hamiltonian is easily calculated to be given by

$$\tilde{H}_n = e^{iV_n} H_n e^{-iV_n} = \epsilon b^* b + (E - \frac{\lambda^2}{\epsilon}) \eta_n . \quad (4.2.5)$$

From the definition of V_n we obtain that

$$\tilde{\eta}_n = e^{iV_n} \eta_n e^{-iV_n} = \eta_n.$$

To compute $\tilde{b} = e^{iV_n} (\mathbb{1} \otimes b) e^{-iV_n}$, we note that if

$$F(\nu) := e^{\nu b^* - \nu b} b e^{-(\nu b^* - \nu b)},$$

then

$$\frac{dF(\nu)}{d\nu} = e^{\nu b^*} [b^*, b] e^{-\nu b^*} = -1.$$

Therefore

$$F(\nu) = F(0) - \nu. \tag{4.2.6}$$

Applying this formula to \tilde{b} we find that

$$\tilde{b} = b \otimes \mathbb{1} - \mathbb{1} \otimes \frac{\lambda}{\epsilon} \eta_n,$$

and similarly

$$\tilde{b}^* = b^* \otimes \mathbb{1} - \mathbb{1} \otimes \frac{\lambda}{\epsilon} \eta_n.$$

Note that the dynamics generated by \tilde{H}_n (or by H_n) leaves the atomic operator η_n invariant

$$e^{i\tau \tilde{H}_n} \eta_n e^{-i\tau \tilde{H}_n} = \eta_n.$$

Lemma 2. *For any state ρ on \mathcal{H}_C*

$$\begin{aligned} \mathcal{L}(\rho) &= p e^{-\lambda(b^* - b)/\epsilon} e^{-i\tau \epsilon b^* b} e^{\lambda(b^* - b)/\epsilon} \rho e^{-\lambda(b^* - b)/\epsilon} e^{i\tau \epsilon b^* b} e^{\lambda(b^* - b)/\epsilon} \\ &+ (1 - p) e^{-i\tau \epsilon b^* b} \rho e^{i\tau \epsilon b^* b}. \end{aligned} \tag{4.2.7}$$

Proof. Using the shift transformation (4.2.4) \mathcal{L} can be written in the form

$$\mathcal{L}(\rho) = \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}} [e^{-iV_n} e^{-i\tau\tilde{H}_n} e^{iV_n} (\rho \otimes \rho_n) e^{-iV_n} e^{i\tau\tilde{H}_n} e^{iV_n}]. \quad (4.2.8)$$

From definition of V_n and the fact that η_n commutes with $\rho_{\mathcal{A}_n}$ we have

$$e^{iV_n} (\rho \otimes \rho_n) e^{-iV_n} = e^{\lambda(b^*-b)/\epsilon} \rho e^{-\lambda(b^*-b)/\epsilon} \otimes \eta_n \rho_n + \rho \otimes (1 - \eta_n) \rho_n. \quad (4.2.9)$$

Therefore, plugging this expression into (4.2.8) we obtain

$$\begin{aligned} \mathcal{L}(\rho) = & \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}} [e^{-iV_n} e^{-i\tau\tilde{H}_n} (e^{\lambda(b^*-b)/\epsilon} \rho e^{-\lambda(b^*-b)/\epsilon} \otimes \eta_n \rho_n) e^{i\tau\tilde{H}_n} e^{iV_n}] \\ & + \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}} [e^{-iV_n} e^{-i\tau\tilde{H}_n} (\rho \otimes (1 - \eta_n) \rho_n) e^{i\tau\tilde{H}_n} e^{iV_n}]. \end{aligned}$$

Using the diagonal form (4.2.5) of the Hamiltonian, we have

$$\begin{aligned} \mathcal{L}(\rho) = & \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}} [e^{-\lambda(b^*-b)/\epsilon} e^{-i\tau\epsilon b^*b} e^{\lambda(b^*-b)/\epsilon} \rho e^{-\lambda(b^*-b)/\epsilon} e^{i\tau\epsilon b^*b} e^{\lambda(b^*-b)/\epsilon} \otimes \eta_n \rho_n] \\ & + \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}} [e^{-i\tau\epsilon b^*b} e^{\lambda(b^*-b)/\epsilon} \rho e^{-\lambda(b^*-b)/\epsilon} e^{i\tau\epsilon b^*b} \otimes (1 - \eta_n) \eta_n \rho_n] \\ & + \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}} [e^{-\lambda(b^*-b)/\epsilon} e^{-i\tau\epsilon b^*b} \rho e^{i\tau\epsilon b^*b} e^{\lambda(b^*-b)/\epsilon} \otimes \eta_n (1 - \eta_n) \rho_n] \\ & + \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}} [e^{-i\tau\epsilon b^*b} \rho e^{i\tau\epsilon b^*b} \otimes (1 - \eta_n) \rho_n]. \end{aligned}$$

Since $(1 - \eta_n)\eta_n = 0$, we get

$$\begin{aligned} \mathcal{L}(\rho) = & \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}} [e^{-\lambda(b^*-b)/\epsilon} e^{-i\tau\epsilon b^*b} e^{\lambda(b^*-b)/\epsilon} \rho e^{-\lambda(b^*-b)/\epsilon} e^{i\tau\epsilon b^*b} e^{\lambda(b^*-b)/\epsilon} \otimes \eta_n \rho_n] \\ & + \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}} [e^{-i\tau\epsilon b^*b} \rho e^{i\tau\epsilon b^*b} \otimes (1 - \eta_n) \rho_n] \\ = & p e^{-\lambda(b^*-b)/\epsilon} e^{-i\tau\epsilon b^*b} e^{\lambda(b^*-b)/\epsilon} \rho e^{-\lambda(b^*-b)/\epsilon} e^{i\tau\epsilon b^*b} e^{\lambda(b^*-b)/\epsilon} \\ & + (1 - p) e^{-i\tau\epsilon b^*b} \rho e^{i\tau\epsilon b^*b}. \end{aligned}$$

□

To calculate the expectation of the photon-number operator $\widehat{N} = b^*b$ at $t = n\tau$, we would need to find the action of the n -th power $\mathcal{L}^n(\rho)$ of the operator (4.1.16). But in fact it is easier to calculate this mean value using the n -th power of the adjoint operator \mathcal{L}^* , which for any bounded operator $B \in \mathcal{B}(\mathcal{H}_C)$ and $\rho \in \mathfrak{C}_1(\mathcal{H}_C)$, is defined by relation

$$\mathrm{Tr}_{\mathcal{H}_C}(\mathcal{L}^*(B)\rho) := \mathrm{Tr}_{\mathcal{H}_C}(B \mathcal{L}(\rho)) . \quad (4.2.10)$$

Using the property (4.1.14) one can obtain explicit an expression for the operator \mathcal{L}^* . Indeed, by (4.2.10) and by (4.1.16)

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}_C}(B \mathcal{L}(\rho)) &= \mathrm{Tr}_{\mathcal{H}_C \otimes \mathcal{H}_{A_n}} \{ (B \otimes \mathbb{1}) e^{-i\tau H_n} (\rho \otimes \mathbb{1}) (\mathbb{1} \otimes \rho_n) e^{i\tau H_n} \} \\ &= \mathrm{Tr}_{\mathcal{H}_C \otimes \mathcal{H}_{A_n}} \{ (\mathbb{1} \otimes \rho_n) e^{-i\tau H_n} (B \otimes \mathbb{1}) e^{i\tau H_n} (\rho \otimes \mathbb{1}) \} \\ &= \mathrm{Tr}_{\mathcal{H}_C \otimes \mathcal{H}_{A_n}} \{ e^{i\tau H_n} (B \otimes \rho_n) e^{-i\tau H_n} (\rho \otimes \mathbb{1}) \} , \end{aligned} \quad (4.2.11)$$

where we used the commutator (4.1.14) and the cyclicity of the trace $\mathrm{Tr}_{\mathcal{H}_C \otimes \mathcal{H}_{A_n}}$. Then (4.2.11) yields

$$\mathcal{L}^*(B) = \mathrm{Tr}_{\mathcal{H}_{A_n}} \{ e^{i\tau H_n} (B \otimes \rho_n) e^{-i\tau H_n} \} , \quad (4.2.12)$$

which according to (4.1.16), is independent of $n \geq 1$.

Remark 1. For density matrix $\rho \in \mathcal{C}_1(\mathcal{H}_C)$ such that $\mathrm{Tr}_{\mathcal{H}_C}(\mathcal{P}(b, b^*) \mathcal{L}(\rho)) < \infty$ for any polynomial $\mathcal{P}(b, b^*)$, one can extend the definition (4.2.10) of \mathcal{L}^* to this class of unbounded observables $\mathfrak{A}(\mathcal{H}_C)$. The advantage of using the dual operator \mathcal{L}^* is that its consecutive application does not increase the degree of the polynomials generated by operators b and b^* .

Now following the line of reasoning of Lemma 2 we deduce from (4.2.12) that

$$\begin{aligned} \mathcal{L}^*(B) &= p e^{-\lambda(b^*-b)/\epsilon} e^{i\tau\epsilon b^*b} e^{\lambda(b^*-b)/\epsilon} B e^{-\lambda(b^*-b)/\epsilon} e^{-i\tau\epsilon b^*b} e^{\lambda(b^*-b)/\epsilon} \\ &+ (1-p) e^{-i\tau\epsilon b^*b} B e^{i\tau\epsilon b^*b} . \end{aligned} \quad (4.2.13)$$

Lemma 3. *For $B = b^*b$ and for the adjoint operator \mathcal{L}^* defined by (4.2.13) we obtain:*

$$\begin{aligned} (\mathcal{L}^*)^n(b^*b) &= b^*b + p \frac{\lambda}{\epsilon} [(1 - e^{ni\epsilon\tau}) b^* + (1 - e^{-ni\epsilon\tau}) b] \\ &+ n p (1-p) \frac{2\lambda^2}{\epsilon^2} (1 - \cos \epsilon\tau) + p^2 \frac{2\lambda^2}{\epsilon^2} (1 - \cos n\epsilon\tau) . \end{aligned} \quad (4.2.14)$$

Proof. From (4.2.6) one gets

$$e^{\lambda(b^*-b)/\epsilon} b^\# e^{-\lambda(b^*-b)/\epsilon} = b^\# - \lambda/\epsilon, \text{ where } b^\# = b \text{ or } b^*.$$

And therefore

$$e^{\lambda(b^*-b)/\epsilon} b^* b e^{-\lambda(b^*-b)/\epsilon} = (b^* - \lambda/\epsilon)(b - \lambda/\epsilon).$$

From the CCR properties of b and b^* we find that their evolution is

$$\begin{aligned} e^{i\tau\epsilon b^*b} b e^{-i\tau\epsilon b^*b} &= e^{-i\tau\epsilon} b \\ e^{i\tau\epsilon b^*b} b^* e^{-i\tau\epsilon b^*b} &= e^{i\tau\epsilon} b^* . \end{aligned}$$

Therefore, making the shift to calculate the first term in (4.2.13) we get

$$\begin{aligned} &e^{-\lambda(b^*-b)/\epsilon} (e^{i\tau\epsilon} b^* - \lambda/\epsilon) (e^{-i\tau\epsilon} b - \lambda/\epsilon) e^{\lambda(b^*-b)/\epsilon} \\ &= (e^{i\tau\epsilon} b^* - (1 - e^{i\tau\epsilon}) \lambda/\epsilon) (e^{-i\tau\epsilon} b - (1 - e^{-i\tau\epsilon}) \lambda/\epsilon) . \end{aligned} \quad (4.2.15)$$

Hence, (4.2.13) and (4.2.15) imply

$$\begin{aligned}
\mathcal{L}^*(b^*b) &= p e^{-\lambda(b^*-b)/\epsilon} e^{i\tau\epsilon b^*b} e^{\lambda(b^*-b)/\epsilon} b^*b e^{-\lambda(b^*-b)/\epsilon} e^{-i\tau\epsilon b^*b} e^{\lambda(b^*-b)/\epsilon} \\
&\quad + (1-p) e^{-i\tau\epsilon b^*b} b^*b e^{i\tau\epsilon b^*b} \\
&= p(e^{i\epsilon\tau}b^* - (1 - e^{i\epsilon\tau})\lambda/\epsilon)(e^{-i\epsilon\tau}b - (1 - e^{-i\epsilon\tau})\lambda/\epsilon) + (1-p)b^*b \\
&= b^*b + p\frac{\lambda}{\epsilon}(1 - e^{i\epsilon\tau})b^* + p\frac{\lambda}{\epsilon}(1 - e^{-i\epsilon\tau})b + p\frac{2\lambda^2}{\epsilon^2}(1 - \cos \epsilon\tau) .
\end{aligned} \tag{4.2.16}$$

If in (4.2.13) we put $B = b^*$, then one gets

$$\mathcal{L}^*(b^*) = e^{i\epsilon\tau}b^* - p(1 - e^{i\epsilon\tau})\lambda/\epsilon , \tag{4.2.17}$$

and similarly, one obtains:

$$\mathcal{L}^*(b) = e^{-i\epsilon\tau}b - p(1 - e^{-i\epsilon\tau})\lambda/\epsilon , \tag{4.2.18}$$

for $B = b$.

Now we are going to prove the n -th power formula (4.2.14) by induction. Note that if we take $n = 1$ in this formula we get (4.2.16).

Suppose that (4.2.14) is true for n to check that it is also valid for $n + 1$.

$$\begin{aligned}
(\mathcal{L}^*)^{n+1}(b^*b) &= \mathcal{L}^*((\mathcal{L}^*)^n(b^*b)) \\
&= \mathcal{L}^*(b^*b) + p\frac{\lambda}{\epsilon}(1 - e^{ni\epsilon\tau})\mathcal{L}^*(b^*) + p\frac{\lambda}{\epsilon}(1 - e^{-ni\epsilon\tau})\mathcal{L}^*(b) \\
&\quad + p\frac{2\lambda^2}{\epsilon^2}n(1 - \cos \epsilon\tau) - p^2\frac{2\lambda^2}{\epsilon^2}n(1 - \cos \epsilon\tau) + p^2\frac{2\lambda^2}{\epsilon^2}(1 - \cos n\epsilon\tau).
\end{aligned}$$

Taking onto account (4.2.16), (4.2.17) and (4.2.18) we can express the action of

operator \mathcal{L}^* as follows

$$\begin{aligned}
(\mathcal{L}^*)^{n+1}(b^*b) &= b^*b + p\frac{\lambda}{\epsilon}(1 - e^{i\epsilon\tau})b^* + p\frac{\lambda}{\epsilon}(1 - e^{-i\epsilon\tau})b + p\frac{2\lambda^2}{\epsilon^2}(1 - \cos \epsilon\tau) \\
&\quad + p\frac{\lambda}{\epsilon}(1 - e^{ni\epsilon\tau})e^{i\epsilon\tau}b^* - p^2\frac{\lambda^2}{\epsilon^2}(1 - e^{i\epsilon\tau})(1 - e^{ni\epsilon\tau}) \\
&\quad + p\frac{\lambda}{\epsilon}(1 - e^{-ni\epsilon\tau})e^{-i\epsilon\tau}b - p^2\frac{\lambda^2}{\epsilon^2}(1 - e^{-i\epsilon\tau})(1 - e^{-ni\epsilon\tau}) \\
&\quad + p\frac{2\lambda^2}{\epsilon^2}n(1 - \cos \epsilon\tau) - p^2\frac{2\lambda^2}{\epsilon^2}n(1 - \cos \epsilon\tau) + p^2\frac{2\lambda^2}{\epsilon^2}(1 - \cos n\epsilon\tau) \\
&= b^*b + p\frac{\lambda}{\epsilon}(1 - e^{(n+1)i\epsilon\tau})b^* + p\frac{\lambda}{\epsilon}(1 - e^{-(n+1)i\epsilon\tau})b \\
&\quad + p\frac{2\lambda^2}{\epsilon^2}(1 - \cos \epsilon\tau)(n+1) - p^2\frac{\lambda^2}{\epsilon^2}(2 - 2\cos \epsilon\tau - 2\cos n\epsilon\tau) \\
&\quad + 2\cos(n+1)\epsilon\tau + 2n - 2n\cos \epsilon\tau - 2 - 2\cos n\epsilon\tau.
\end{aligned}$$

Simplifying the last expression we get

$$\begin{aligned}
(\mathcal{L}^*)^{n+1}(b^*b) &= b^*b + p\frac{\lambda}{\epsilon}(1 - e^{(n+1)i\epsilon\tau})b^* + p\frac{\lambda}{\epsilon}(1 - e^{-(n+1)i\epsilon\tau})b \\
&\quad + p\frac{2\lambda^2}{\epsilon^2}(1 - \cos \epsilon\tau)(n+1) - p^2\frac{2\lambda^2}{\epsilon^2}(n+1)(1 - \cos \epsilon\tau) \\
&\quad + p^2\frac{2\lambda^2}{\epsilon^2}(1 - \cos(n+1)\epsilon\tau),
\end{aligned}$$

which proves (4.2.14) formula. \square

Proof. (of Theorem 7) To find the number of photons in the cavity at time $t = n\tau$ we take the expectation in (4.2.14). Since the initial state is gauge invariant we get

$$\begin{aligned}
N(t) &= \text{Tr}_{\mathcal{H}_c}(b^*b\mathcal{L}^n(\rho_c^\beta)) = \text{Tr}_{\mathcal{H}_c}((\mathcal{L}^*)^n(b^*b)\rho_c^\beta) \\
&= N(0) + n p(1-p) \frac{2\lambda^2}{\epsilon^2} (1 - \cos \epsilon\tau) + p^2 \frac{2\lambda^2}{\epsilon^2} (1 - \cos n\epsilon\tau) .
\end{aligned}$$

\square

Remark 2. *Theorem implies that only flux of randomly exited atoms (i.e. $0 < p < 1$) is able to produce a pumping of the cavity by photons. Since the number of photons in cavity unboundedly increases with time, there is no regular limiting state.*

4.3 Number of photons in leaking cavity

In the Hamiltonian case the number of photons in the cavity increases in time. To stabilize the energy in the cavity we consider the case when the generator of the dynamics (4.1.21) has a nonzero dissipative part, which describes the leaking of photons out of the cavity. So suppose that $\sigma > 0$.

The next theorem shows that the number of photons in the leaking cavity stabilizes in time for any $\sigma > 0$. It is obviously different to the case $\sigma = 0$, see Theorem 7.

Theorem 8. *For any $\sigma > 0$ and arbitrary gauge-invariant initial cavity state ρ_C such that*

$$\omega_C^t(b^*b) |_{t=0} = \text{Tr}_{\mathcal{H}_C}(b^*b \rho_C) < \infty , \quad (4.3.1)$$

one gets for the limit of the cavity photon-number mean value

$$\begin{aligned} N_\sigma^\infty &= \omega_C(b^*b) := \lim_{t \rightarrow \infty} \omega_C^t(b^*b) = \\ &= \frac{4\lambda^2}{4\epsilon^2 + \sigma^2} \frac{p}{1 - e^{-\sigma\tau}} \{1 + e^{-\sigma\tau}(1 - 2p) - 2e^{-\sigma\tau/2}(1 - p) \cos \epsilon\tau\} . \end{aligned} \quad (4.3.2)$$

Lemma 4. *For any state ρ on algebra $\mathfrak{A}(\mathcal{H}_C)$ one has:*

$$\mathcal{L}_\sigma(\rho) = p e^{-\lambda(b^*-b)/\epsilon} e^{\tau L_\lambda^C} (e^{\lambda(b^*-b)/\epsilon} \rho e^{-\lambda(b^*-b)/\epsilon}) e^{\lambda(b^*-b)/\epsilon} + (1 - p) e^{\tau L_0^C}(\rho),$$

where L_λ^C acts on $\mathfrak{A}(\mathcal{H}_C)$ as follows

$$L_\lambda^C(\rho) := -i[\epsilon b^* b, \rho] + \sigma(b - \lambda/\epsilon)\rho(b^* - \lambda/\epsilon) - \frac{\sigma}{2}\{(b^* - \lambda/\epsilon)(b - \lambda/\epsilon), \rho\}. \quad (4.3.3)$$

Proof. Using unitary transformation generated by (4.2.4) of the Hamiltonian, we define instead of (4.1.23):

$$\begin{aligned} \tilde{L}_{\sigma,n}(\rho \otimes \rho_n) &:= e^{iV_n} L_{\sigma,n}(e^{-iV_n}(\rho \otimes \rho_n)e^{iV_n})e^{-iV_n} \\ &= -i[\tilde{H}_n, \rho \otimes \rho_n] + \sigma e^{iV_n} b e^{-iV_n}(\rho \otimes \rho_n) e^{iV_n} b^* e^{-iV_n} - \frac{\sigma}{2}\{e^{iV_n} b^* b e^{-iV_n}, \rho \otimes \rho_n\} \\ &= -i[\epsilon b^* b + (E - \frac{\lambda^2}{\epsilon})\eta_n, \rho \otimes \rho_n] + [\sigma(b - \lambda/\epsilon)\rho(b^* - \lambda/\epsilon) \\ &\quad - \frac{\sigma}{2}\{(b^* - \lambda/\epsilon)(b - \lambda/\epsilon), \rho\}] \otimes \eta_n \rho_n + (\sigma b^* \rho b - \frac{\sigma}{2}\{b^* b, \rho\}) \otimes (1 - \eta_n) \rho_n \\ &= L_\lambda^C(\rho) \otimes \eta_n \rho_n + L_0^C(\rho) \otimes (1 - \eta_n) \rho_n, \end{aligned} \quad (4.3.4)$$

here $L_0^C := L_{\lambda=0}^C$, see (4.3.3). Then discrete evolution operator (4.1.25) gets the form

$$\begin{aligned} \mathcal{L}_\sigma(\rho) &= \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}}(e^{\tau L_{\sigma,n}}(\rho \otimes \rho_n)) = \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}}(e^{-iV_n} e^{\tau \tilde{L}_{\sigma,n}}(e^{iV_n}(\rho \otimes \rho_n)e^{-iV_n})e^{iV_n}) \\ &= \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}}(e^{-iV_n} e^{\tau \tilde{L}_{\sigma,n}}(e^{\lambda(b^*-b)/\epsilon} \rho e^{-\lambda(b^*-b)/\epsilon} \otimes \eta_n \rho_n + \rho \otimes (1 - \eta_n) \rho_n) e^{iV_n}) \\ &= \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}}(e^{-iV_n} (e^{\tau L_\lambda^C}(e^{\lambda(b^*-b)/\epsilon} \rho e^{-\lambda(b^*-b)/\epsilon}) \otimes \eta_n \rho_n) e^{iV_n}) \\ &\quad + \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}}(e^{-iV_n} (e^{\tau L_0^C}(\rho) \otimes (1 - \eta_n) \rho_n) e^{iV_n}). \end{aligned}$$

Simplifying the last expression

$$\begin{aligned}
\mathcal{L}_\sigma(\rho) &= \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}}(e^{-\lambda(b^*-b)/\epsilon} e^{\tau L_\lambda^C} (e^{\lambda(b^*-b)/\epsilon} \rho e^{-\lambda(b^*-b)/\epsilon}) e^{\lambda(b^*-b)/\epsilon} \otimes \eta_n \rho_n) \\
&\quad + \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}}(e^{\tau L_0^C}(\rho) \otimes (1 - \eta_n) \rho_n) \\
&= p e^{-\lambda(b^*-b)/\epsilon} e^{\tau L_\lambda^C} (e^{\lambda(b^*-b)/\epsilon} \rho e^{-\lambda(b^*-b)/\epsilon}) e^{\lambda(b^*-b)/\epsilon} + (1 - p) e^{\tau L_0^C}(\rho).
\end{aligned}$$

□

Now we look for the corresponding adjoint operator \mathcal{L}_σ^* . It can be calculated using the shift transformation (4.3.4) and the adjoint operator (4.2.12). For any bounded operator $B \in \mathcal{B}(\mathcal{H}_C)$ one has:

$$\begin{aligned}
\mathcal{L}_\sigma^*(B) &= \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}}(e^{\tau L_n^*}(B \otimes \rho_n)) \\
&= p e^{-\lambda(b^*-b)/\epsilon} e^{\tau (L_\lambda^C)^*} e^{\lambda(b^*-b)/\epsilon} B e^{-\lambda(b^*-b)/\epsilon} e^{\lambda(b^*-b)/\epsilon} + (1 - p) e^{\tau (L_0^C)^*}(B).
\end{aligned} \tag{4.3.5}$$

Here the adjoint operators $(L_\lambda^C)^*$ and $(L_0^C)^*$ act as follows:

$$(L_\lambda^C)^*(B) = i[\epsilon b^* b, B] + \sigma(b^* - \lambda/\epsilon)B(b - \lambda/\epsilon) - \frac{\sigma}{2}\{(b^* - \lambda/\epsilon)(b - \lambda/\epsilon), B\} \tag{4.3.6}$$

$$(L_0^C)^*(B) = (\tilde{L}_C^0)^*(B) = i[\epsilon b^* b, B] + \sigma b^* B b - \frac{\sigma}{2}\{b^* b, B\} \tag{4.3.7}$$

Remark 3. As we indicated in Remark 1, one can extend the adjoint operator \mathcal{L}_σ^* to the algebra of polynomial observables $\mathcal{P}(b, b^*) \in \mathfrak{A}(\mathcal{H}_C)$.

The proof of the Theorem 8 is similar to the case $\sigma = 0$. The number of photons for the time $t = n\tau$ can be calculated using the adjoint operator

$$N_\sigma^{t=n\tau} := \omega_C^t(b^* b) = \text{Tr}_{\mathcal{H}_C}(b^* b \mathcal{L}_\sigma^n(\rho_C)) = \text{Tr}_{\mathcal{H}_C}((\mathcal{L}_\sigma^*)^n(b^* b) \rho_C), \tag{4.3.8}$$

where ρ_C is a gauge-invariant state of the cavity, see Remark 3.

We look more closely of the adjoint operator \mathcal{L}_σ^* . Note that if one takes $B = b^*b$ in (4.3.5) we get

$$\mathcal{L}_\sigma^*(b^*b) = pe^{-\lambda(b^*-b)/\epsilon}e^{\tau(L_\lambda^C)^*}((b^* - \lambda/\epsilon)(b - \frac{\lambda}{\epsilon}))e^{\lambda(b^*-b)/\epsilon} + (1-p)e^{\tau(L_0^C)^*}(b^*b). \quad (4.3.9)$$

Lemma 5. *The action of the adjoint operator \mathcal{L}_σ^* on the number operator of photons can be calculated explicitly*

$$\begin{aligned} \mathcal{L}_\sigma^*(b^*b) = & e^{-\sigma\tau}b^*b + p\frac{i\lambda}{\mu}e^{-\sigma\tau}(1 - e^{\mu\tau})b^* - p\frac{i\lambda}{\bar{\mu}}e^{-\sigma\tau}(1 - e^{\bar{\mu}\tau})b \\ & + p\frac{\lambda^2}{|\mu|^2}e^{-\sigma\tau}(1 - e^{\mu\tau})(1 - e^{\bar{\mu}\tau}). \end{aligned} \quad (4.3.10)$$

Proof. Consider the first term in (4.3.9). Let $\gamma_{\lambda,\tau}(B) = e^{\tau(L_\lambda^C)^*}(B)$, for $B \in \mathfrak{A}(\mathcal{H}_C)$.

To find $\gamma_{\lambda,\tau}((b^* - \lambda/\epsilon)(b - \lambda/\epsilon))$ we first note that

$$(L_\lambda^C)^*((b^* - \lambda/\epsilon)(b - \lambda/\epsilon)) = i\lambda b - i\lambda b^* - \sigma(b^* - \lambda/\epsilon)(b - \lambda/\epsilon)$$

and

$$(L_\lambda^C)^*(b) = -i\epsilon b - \frac{\sigma}{2}b + \frac{\sigma\lambda}{2\epsilon} = -\mu b + \frac{\sigma\lambda}{2\epsilon}$$

and

$$(L_\lambda^C)^*(b^*) = i\epsilon b^* - \frac{\sigma}{2}b^* + \frac{\sigma\lambda}{2\epsilon} = -\bar{\mu}b^* + \frac{\sigma\lambda}{2\epsilon},$$

where $\mu = \frac{\sigma}{2} + i\epsilon$. Therefore we have the following system of differential equations

for $\gamma_{\lambda,\tau}$

$$\begin{aligned}\frac{d\gamma_{\lambda,\tau}((b^* - \lambda/\epsilon)(b - \lambda/\epsilon))}{d\tau} &= -\sigma\gamma_{\lambda,\tau}((b^* - \lambda/\epsilon)(b - \lambda/\epsilon)) + i\lambda\gamma_{\lambda,\tau}(b) - i\lambda\gamma_{\lambda,\tau}(b^*) \\ \frac{d\gamma_{\lambda,\tau}(b)}{d\tau} &= -\mu\gamma_{\lambda,\tau}(b) + \frac{\lambda\sigma}{2\epsilon} \\ \frac{d\gamma_{\lambda,\tau}(b^*)}{d\tau} &= -\bar{\mu}\gamma_{\lambda,\tau}(b^*) + \frac{\lambda\sigma}{2\epsilon}.\end{aligned}$$

The solution to this system is

$$\gamma_{\lambda,\tau}(b) = e^{-\mu\tau}(b - \frac{\lambda\sigma}{2\epsilon\mu}) + \frac{\lambda\sigma}{2\epsilon\mu} = e^{-\mu\tau}b + \frac{\lambda\sigma}{2\epsilon\mu}(1 - e^{-\mu\tau}) \quad (4.3.11)$$

$$\gamma_{\lambda,\tau}(b^*) = e^{-\bar{\mu}\tau}(b^* - \frac{\lambda\sigma}{2\epsilon\bar{\mu}}) + \frac{\lambda\sigma}{2\epsilon\bar{\mu}} = e^{-\bar{\mu}\tau}b^* + \frac{\lambda\sigma}{2\epsilon\bar{\mu}}(1 - e^{-\bar{\mu}\tau}) \quad (4.3.12)$$

$$\begin{aligned}\gamma_{\lambda,\tau}((b^* - \lambda/\epsilon)(b - \lambda/\epsilon)) &= e^{-\sigma\tau}b^*b - \frac{\lambda}{\epsilon}b^*e^{-\sigma\tau}(\frac{i\epsilon}{\mu}e^{\mu\tau} - \frac{i\epsilon}{\mu} + 1) \\ &+ \lambda/\epsilon b e^{-\sigma\tau}(\frac{i\epsilon}{\bar{\mu}}e^{\bar{\mu}\tau} - \frac{i\epsilon}{\bar{\mu}} - 1) + \frac{\lambda^2}{|\mu|^2}(1 - e^{-\sigma\tau}) + \frac{\lambda^2}{\epsilon^2}e^{-\sigma\tau} - \frac{\lambda^2\sigma \sin \epsilon\tau}{\epsilon|\mu|^2}e^{-\frac{\sigma}{2}\tau}.\end{aligned} \quad (4.3.13)$$

Making the shift transformation $\gamma_{\lambda,\tau}((b^* - \lambda/\epsilon)(b - \lambda/\epsilon))$ we get

$$\begin{aligned}e^{-\lambda(b^*-b)/\epsilon}\gamma_{\lambda,\tau}((b^* - \lambda/\epsilon)(b - \lambda/\epsilon))e^{\lambda(b^*-b)/\epsilon} &= e^{-\sigma\tau}(b^* + \lambda/\epsilon)(b + \lambda/\epsilon) \\ &- \lambda/\epsilon(b^* + \lambda/\epsilon)e^{-\sigma\tau}(\frac{i\epsilon}{\mu}e^{\mu\tau} - \frac{i\epsilon}{\mu} + 1) \\ &+ \lambda/\epsilon(b + \lambda/\epsilon)e^{-\sigma\tau}(\frac{i\epsilon}{\bar{\mu}}e^{\bar{\mu}\tau} - \frac{i\epsilon}{\bar{\mu}} - 1) + \frac{\lambda^2}{|\mu|^2}(1 - e^{-\sigma\tau}) + \frac{\lambda^2}{\epsilon^2}e^{-\sigma\tau} - \frac{\lambda^2\sigma \sin \epsilon\tau}{\epsilon|\mu|^2}e^{-\frac{\sigma}{2}\tau} \\ &= e^{-\sigma\tau}b^*b + \frac{i\lambda}{\mu}e^{-\sigma\tau}(1 - e^{i\mu\tau})b^* - \frac{i\lambda}{\bar{\mu}}e^{-\sigma\tau}(1 - e^{-i\bar{\mu}\tau})b + \frac{\lambda^2}{|\mu|^2}(1 + e^{-\sigma\tau} - 2e^{-\frac{\sigma}{2}\tau} \cos \epsilon\tau).\end{aligned}$$

Calculating the second term in (4.3.9) by setting $\lambda = 0$, we obtain

$$\begin{aligned}\mathcal{L}_\sigma^*(b^*b) = & e^{-\sigma\tau}b^*b + p\frac{i\lambda}{\mu}e^{-\sigma\tau}(1 - e^{\mu\tau})b^* - p\frac{i\lambda}{\bar{\mu}}e^{-\sigma\tau}(1 - e^{\bar{\mu}\tau})b \\ & + p\frac{\lambda^2}{|\mu|^2}e^{-\sigma\tau}(1 - e^{\mu\tau})(1 - e^{\bar{\mu}\tau}).\end{aligned}$$

□

Note that (4.3.11) and (4.3.12) yield

$$\mathcal{L}_\sigma^*(b^*) = e^{i\epsilon\tau}e^{-\frac{\sigma}{2}\tau}b^* + p\frac{2i\lambda}{\sigma - 2i\epsilon}(1 - e^{i\epsilon\tau}e^{-\frac{\sigma}{2}\tau}) = e^{-\bar{\mu}\tau}b^* + p\frac{i\lambda}{\bar{\mu}}(1 - e^{-\bar{\mu}\tau}) \quad (4.3.14)$$

and

$$\mathcal{L}_\sigma^*(b) = e^{-i\epsilon\tau}e^{-\frac{\sigma}{2}\tau}b - p\frac{2i\lambda}{\sigma + 2i\epsilon}(1 - e^{-i\epsilon\tau}e^{-\frac{\sigma}{2}\tau}) = e^{-\mu\tau}b - p\frac{i\lambda}{\mu}(1 - e^{-\mu\tau}). \quad (4.3.15)$$

Proof. (of Theorem 8) The following expression can be found by representing the operator \mathcal{L}_σ^* in the matrix form

$$\begin{aligned}(\mathcal{L}_\sigma^*)^n(b^*b) = & e^{-n\sigma\tau}b^*b + p\frac{i\lambda}{\mu}(e^{-n\sigma\tau} - e^{-n\bar{\mu}\tau})b^* - p\frac{i\lambda}{\bar{\mu}}(e^{-n\sigma\tau} - e^{-n\mu\tau})b \quad (4.3.16) \\ & + p\frac{\lambda^2}{|\mu|^2}e^{-\sigma\tau}(1 - e^{\mu\tau})(1 - e^{\bar{\mu}\tau})\frac{1 - e^{-n\sigma\tau}}{1 - e^{-\sigma\tau}} - p^2\frac{2\lambda^2}{|\mu|^2}\frac{1 - e^{-n\sigma\tau}}{1 - e^{-\sigma\tau}}(1 - e^{-\frac{\sigma}{2}\tau}\cos\epsilon\tau) \\ & + p^2\frac{2\lambda^2}{|\mu|^2}(1 - e^{-n\frac{\sigma}{2}\tau}\cos n\epsilon\tau).\end{aligned}$$

We prove this formula by induction.

Suppose (4.3.16) is true for n , we show it is true for $n + 1$.

$$\begin{aligned}
& (\mathcal{L}_\sigma^*)^{n+1}(b^*b) \\
&= e^{-n\sigma\tau} \mathcal{L}_\sigma^*(b^*b) + p \frac{i\lambda}{\mu} (e^{-n\sigma\tau} - e^{-n\bar{\mu}\tau}) \mathcal{L}_\sigma^*(b^*) - p \frac{i\lambda}{\bar{\mu}} (e^{-n\sigma\tau} - e^{-n\mu\tau}) \mathcal{L}_\sigma^*(b) \\
&+ p \frac{\lambda^2}{|\mu|^2} e^{-\sigma\tau} (1 - e^{\mu\tau}) (1 - e^{\bar{\mu}\tau}) \frac{1 - e^{-n\sigma\tau}}{1 - e^{-\sigma\tau}} - p^2 \frac{2\lambda^2}{|\mu|^2} \frac{1 - e^{-n\sigma\tau}}{1 - e^{-\sigma\tau}} (1 - e^{-\frac{\sigma}{2}\tau} \cos \epsilon\tau) \\
&+ p^2 \frac{2\lambda^2}{|\mu|^2} (1 - e^{-n\frac{\sigma}{2}\tau} \cos n\epsilon\tau).
\end{aligned}$$

From the formulas (4.3.10), (4.3.14) and (4.3.15) we get

$$\begin{aligned}
& (\mathcal{L}_\sigma^*)^{n+1}(b^*b) \\
&= e^{-n\sigma\tau} (e^{-\sigma\tau} b^*b + p \frac{i\lambda}{\mu} e^{-\sigma\tau} (1 - e^{\mu\tau}) b^* - p \frac{i\lambda}{\bar{\mu}} e^{-\sigma\tau} (1 - e^{\bar{\mu}\tau}) b \\
&+ p \frac{\lambda^2}{|\mu|^2} e^{-\sigma\tau} (1 - e^{\mu\tau}) (1 - e^{\bar{\mu}\tau})) + p \frac{i\lambda}{\mu} (e^{-n\sigma\tau} - e^{-n\bar{\mu}\tau}) e^{-\bar{\mu}\tau} b^* + p \frac{i\lambda}{\bar{\mu}} (1 - e^{-\bar{\mu}\tau}) \\
&- p \frac{i\lambda}{\bar{\mu}} (e^{-n\sigma\tau} - e^{-n\mu\tau}) e^{-\mu\tau} b - p \frac{i\lambda}{\mu} (1 - e^{-\mu\tau}) \\
&+ p \frac{\lambda^2}{|\mu|^2} e^{-\sigma\tau} (1 - e^{\mu\tau}) (1 - e^{\bar{\mu}\tau}) \frac{1 - e^{-n\sigma\tau}}{1 - e^{-\sigma\tau}} - p^2 \frac{2\lambda^2}{|\mu|^2} \frac{1 - e^{-n\sigma\tau}}{1 - e^{-\sigma\tau}} (1 - e^{-\frac{\sigma}{2}\tau} \cos \epsilon\tau) \\
&+ p^2 \frac{2\lambda^2}{|\mu|^2} (1 - e^{-n\frac{\sigma}{2}\tau} \cos n\epsilon\tau).
\end{aligned}$$

Simplifying the latest expression we get

$$\begin{aligned}
& (\mathcal{L}_\sigma^*)^{n+1}(b^*b) \\
&= e^{-(n+1)\sigma\tau} b^*b + p \frac{i\lambda}{\mu} (e^{-(n+1)\sigma\tau} - e^{-(n+1)\bar{\mu}\tau}) b^* - p \frac{i\lambda}{\bar{\mu}} (e^{-(n+1)\sigma\tau} - e^{-(n+1)\mu\tau}) b \\
&+ p \frac{\lambda^2}{|\mu|^2} e^{-\sigma\tau} (1 - e^{\mu\tau}) (1 - e^{\bar{\mu}\tau}) \frac{1 - e^{-(n+1)\sigma\tau}}{1 - e^{-\sigma\tau}} \\
&- p^2 \frac{2\lambda^2}{|\mu|^2} \frac{1}{1 - e^{-\sigma\tau}} (-e^{-\frac{\sigma}{2}\tau} \cos \epsilon\tau + e^{-\sigma\tau} + e^{-(n+1)\frac{\sigma}{2}\tau} \cos(n+1)\epsilon\tau - e^{-(n+1)\sigma\tau} \\
&+ e^{-(n+1)\sigma\tau} e^{-\frac{\sigma}{2}\tau} \cos \epsilon\tau - e^{-(n+1)\frac{\sigma}{2}\tau} e^{-\sigma\tau} \cos(n+1)\epsilon\tau) \\
&= e^{-(n+1)\sigma\tau} b^*b + p \frac{i\lambda}{\mu} (e^{-(n+1)\sigma\tau} - e^{-(n+1)\bar{\mu}\tau}) b^* - p \frac{i\lambda}{\bar{\mu}} (e^{-(n+1)\sigma\tau} - e^{-(n+1)\mu\tau}) b \\
&+ p \frac{\lambda^2}{|\mu|^2} e^{-\sigma\tau} (1 - e^{\mu\tau}) (1 - e^{\bar{\mu}\tau}) \frac{1 - e^{-(n+1)\sigma\tau}}{1 - e^{-\sigma\tau}} \\
&- p^2 \frac{2\lambda^2}{|\mu|^2} \frac{1 - e^{-(n+1)\sigma\tau}}{1 - e^{-\sigma\tau}} (1 - e^{-\frac{\sigma}{2}\tau} \cos \epsilon\tau) + p^2 \frac{2\lambda^2}{|\mu|^2} (1 - e^{-(n+1)\frac{\sigma}{2}\tau} \cos(n+1)\epsilon\tau),
\end{aligned}$$

which proves (4.3.16).

Note that if we take $n = 1$ in (4.3.16) we get (4.3.9), if we take $\sigma = 0$ we get (4.2.14) and if $\tau = 0$ we get b^*b .

In the limit n goes to infinity we get

$$\begin{aligned}
& w^* - \lim_{n \rightarrow \infty} (\mathcal{L}_\sigma^*)^n(b^*b) \\
&= p \frac{\lambda^2}{|\mu|^2} e^{-\sigma\tau} (1 - e^{\mu\tau}) (1 - e^{\bar{\mu}\tau}) \frac{1}{1 - e^{-\sigma\tau}} - p^2 \frac{2\lambda^2}{|\mu|^2} \frac{1}{1 - e^{-\sigma\tau}} (1 - e^{-\frac{\sigma}{2}\tau} \cos \epsilon\tau) + p^2 \frac{2\lambda^2}{|\mu|^2} \\
&= \frac{4\lambda^2}{4\epsilon^2 + \sigma^2} \frac{p}{1 - e^{-\sigma\tau}} \{1 + e^{-\sigma\tau} (1 - 2p) - 2e^{-\sigma\tau/2} (1 - p) \cos \epsilon\tau\}. \tag{4.3.17}
\end{aligned}$$

□

Remark 4. Notice that for $\sigma > 0$ and $p = 0$ the limit value (4.3.2) is zero. This is trivial because any initial cavity state with finite mean-value of photons (4.3.1) will

be exhausted by the leaking, $\sigma > 0$, and the absence of pumping: $p = 0$.

Let $0 < p < 1$. Then for non-resonant case $\epsilon\tau \neq 2\pi s$, where $s \in \mathbb{Z}^1$, the limit

$$\lim_{\sigma \rightarrow 0} \omega_C(b^*b) = +\infty, \quad (4.3.18)$$

which corresponds to conclusion of the Theorem 7 about unlimited pumping of the perfect cavity, i.e. for $\sigma = 0$.

The interpretation of (4.3.2) is less transparent in two special cases:

(a) For the resonant case $\epsilon\tau = 2\pi s$ and $\sigma = 0$ the mean-value of photons (4.2.1) is bounded and equal to $N(0)$ independent of p . Whereas (4.3.2) yields the bound

$$\lim_{\sigma \rightarrow 0} \omega_C(b^*b) = p^2 \frac{\lambda^2}{\epsilon^2}, \quad (4.3.19)$$

which is p -dependent.

(b) For $p = 1$ the mean-value of photons (4.2.1) for the ideal cavity is bounded and oscillates. Although (4.3.2) for $p = 1$ gives

$$\omega_C(b^*b) = \frac{\lambda^2}{|\mu|^2}, \quad (4.3.20)$$

for any leaking $\sigma > 0$. So clearly, the limits: $t \rightarrow \infty$ and $\sigma \rightarrow 0$ do not commute.

4.4 Long-time behavior

Recall that we consider the limiting state (4.1.28) on the Weyl operator algebra. Our result concerning the limiting state of the leaking cavity is the following:

Theorem 9. *Let $\sigma > 0$. For any gauge-invariant initial cavity state ρ_C and for a*

homogenous atomic beam with parameter $p = \text{Tr}_{\mathcal{H}_{\mathcal{A}_n}}(\eta_n \rho_n)$, the limiting cavity state

$$\omega_{\mathcal{C},\sigma}(\cdot) := \lim_{t \rightarrow \infty} \omega_{\mathcal{C},\sigma}^t(\cdot) \quad (4.4.1)$$

exists and it does not depend on $\rho_{\mathcal{C}}$. The explicit form of the limiting functional (4.1.28) on the Weyl operator is

$$\omega_{\mathcal{C},\sigma}(W(\alpha)) = e^{-\frac{|\zeta|^2}{4}} \prod_{k=0}^{\infty} \left\{ p \exp\left\{-\frac{i\lambda}{\mu}(1 - e^{-\mu\tau})e^{-k\mu\tau}\alpha - \frac{i\lambda}{\bar{\mu}}(1 - e^{-\bar{\mu}\tau})e^{k\mu\tau}\bar{\alpha}\right\} + 1 - p \right\}, \quad (4.4.2)$$

where $\mu := i\epsilon + \sigma/2$.

Proof. From Baker-Campbell-Hausdorff formula for the Weyl operators (2.1.4) and from the commutation relations (2.1.5) the action of $(L_{\lambda}^C)^*$ can be expressed as following

$$\begin{aligned} & (L_{\lambda}^C)^*(W(\alpha)) \\ &= i\epsilon[b^*b, W(\alpha)] + \sigma(b^* - \lambda/\epsilon)W(\alpha)(b - \lambda/\epsilon) - \frac{\sigma}{2}\{(b^* - \lambda/\epsilon)(b - \lambda/\epsilon), W(\alpha)\} \\ &= (-\mu\alpha b - \mu|\alpha|^2 + \bar{\mu}\bar{\alpha}b^* + \frac{\lambda\sigma}{2\epsilon}(\alpha - \bar{\alpha}))W(\alpha), \end{aligned} \quad (4.4.3)$$

where we denoted $\mu = \frac{\sigma}{2} + i\epsilon$.

Therefore the dynamics generated by (4.3.6):

$$\gamma_{\lambda,\tau} := e^{\tau(L_{\lambda}^C)^*} \quad (4.4.4)$$

is *quasi-free*, which by definition [22] means that for some $T_{\lambda,\tau}(\alpha)$ and $g_{\lambda,\tau}(\alpha)$

$$\gamma_{\lambda,\tau}(W(\alpha)) = e^{g_{\lambda,\tau}(\alpha)}W(T_{\lambda,\tau}(\alpha)). \quad (4.4.5)$$

See Section 2.2.5 for more details on quasi-free semigroup.

We can find $T_{\lambda,\tau}(\alpha)$ and $g_{\lambda,\tau}(\alpha)$ using the differential equation the dynamics should satisfy

$$\frac{d\gamma_{\lambda,\tau}(W(\alpha))}{d\tau} = (L_\lambda^C)^*(\gamma_{\lambda,\tau}(W(\alpha))). \quad (4.4.6)$$

The right-hand side can be calculated using (4.4.3) equation where instead of α we have $T_{\lambda,\tau}(\alpha)$

$$\begin{aligned} (L_\lambda^C)^*(\gamma_{\lambda,\tau}(W(\alpha))) &= e^{g_{\lambda,\tau}}(\alpha)(-\mu T_{\lambda,\tau}(\alpha)b + \overline{\mu T_{\lambda,\tau}(\alpha)}b^* - \mu|T_{\lambda,\tau}(\alpha)|^2 \\ &+ \frac{\lambda\sigma}{2\epsilon}(T_{\lambda,\tau}(\alpha) - \overline{T_{\lambda,\tau}(\alpha)}))W(T_{\lambda,\tau}(\alpha)) \\ &= (-\mu T_{\lambda,\tau}(\alpha)b + \overline{\mu T_{\lambda,\tau}(\alpha)}b^* - \mu|T_{\lambda,\tau}(\alpha)|^2 + \frac{\lambda\sigma}{2\epsilon}(T_{\lambda,\tau}(\alpha) - \overline{T_{\lambda,\tau}(\alpha)}))\gamma_{\lambda,\tau}(W(\alpha)). \end{aligned} \quad (4.4.7)$$

The derivative of the τ -dependent Weyl operator $W(T_{\lambda,\tau}(\alpha))$ will look like

$$\begin{aligned} \frac{dW(T_{\lambda,\tau}(\alpha))}{d\tau} &= -\frac{d\overline{T_{\lambda,\tau}(\alpha)}}{d\tau}b^*W(T_{\lambda,\tau}(\alpha)) + \frac{dT_{\lambda,\tau}(\alpha)}{d\tau}e^{-\overline{T_{\lambda,\tau}(\alpha)}b^*}be^{T_{\lambda,\tau}b}e^{-\frac{|T_{\lambda,\tau}(\alpha)|^2}{2}} \\ &- \frac{1}{2}\frac{d|T_{\lambda,\tau}(\alpha)|^2}{d\tau}W(T_{\lambda,\tau}(\alpha)) \\ &= \left(\frac{dT_{\lambda,\tau}(\alpha)}{d\tau}b - \frac{d\overline{T_{\lambda,\tau}(\alpha)}}{d\tau}b^* - \frac{1}{2}T_{\lambda,\tau}(\alpha)\frac{d\overline{T_{\lambda,\tau}(\alpha)}}{d\tau} + \frac{1}{2}\overline{T_{\lambda,\tau}(\alpha)}\frac{dT_{\lambda,\tau}(\alpha)}{d\tau} \right) W(T_{\lambda,\tau}(\alpha)) \end{aligned}$$

Therefore $\gamma_{\lambda,\tau}(W(\alpha))$ satisfies the following differential equation

$$\begin{aligned} \frac{d\gamma_{\lambda,\tau}(W(\alpha))}{d\tau} &= e^{g_{\lambda,\tau}(\alpha)} \frac{dg_{\lambda,\tau}(\alpha)}{d\tau} W(T_{\lambda,\tau}(\alpha)) + e^{g_{\lambda,\tau}(\alpha)} \left(\frac{dT_{\lambda,\tau}(\alpha)}{d\tau} b - \frac{\overline{dT_{\lambda,\tau}(\alpha)}}{d\tau} b^* \right. \\ &\quad \left. - \frac{1}{2} T_{\lambda,\tau}(\alpha) \frac{\overline{dT_{\lambda,\tau}(\alpha)}}{d\tau} + \frac{1}{2} \overline{T_{\lambda,\tau}(\alpha)} \frac{dT_{\lambda,\tau}(\alpha)}{d\tau} \right) W(T_{\lambda,\tau}(\alpha)) \\ &= \left(\frac{dT_{\lambda,\tau}(\alpha)}{d\tau} b - \frac{\overline{dT_{\lambda,\tau}(\alpha)}}{d\tau} b^* + \frac{dg_{\lambda,\tau}(\alpha)}{d\tau} - \frac{1}{2} T_{\lambda,\tau}(\alpha) \frac{\overline{dT_{\lambda,\tau}(\alpha)}}{d\tau} + \frac{1}{2} \overline{T_{\lambda,\tau}(\alpha)} \frac{dT_{\lambda,\tau}(\alpha)}{d\tau} \right) \\ &\quad \times \gamma_{\lambda,\tau}(W(\alpha)). \end{aligned} \tag{4.4.8}$$

Because of the differential equation for the dynamics (4.4.6) the last equation (4.4.8) should coincides with the equation (4.4.7). Then we get the following system of differential equations

$$\frac{dT_{\lambda,\tau}(\alpha)}{d\tau} = -\mu T_{\lambda,\tau}(\alpha)$$

and

$$\frac{dg_{\lambda,\tau}(\alpha)}{d\tau} = \frac{1}{2} T_{\lambda,\tau}(\alpha) \frac{\overline{dT_{\lambda,\tau}(\alpha)}}{d\tau} - \frac{1}{2} \overline{T_{\lambda,\tau}(\alpha)} \frac{dT_{\lambda,\tau}(\alpha)}{d\tau} - \mu |T_{\lambda,\tau}(\alpha)|^2 + \frac{\lambda\sigma}{2\epsilon} (T_{\lambda,\tau}(\alpha) - \overline{T_{\lambda,\tau}(\alpha)}).$$

Using the first equation the second one could be simplified and will look like

$$\frac{dg_{\lambda,\tau}(\alpha)}{d\tau} = \frac{\lambda\sigma}{2\epsilon} (T_{\lambda,\tau}(\alpha) - \overline{T_{\lambda,\tau}(\alpha)}) - \frac{\sigma}{2} |T_{\lambda,\tau}(\alpha)|^2.$$

The solution to the first differential equation is

$$T_{\lambda,\tau}(\alpha) = e^{-\mu\tau} \alpha.$$

Therefore the second differential equation can be written as follows

$$\frac{dg_{\lambda,\tau}(\alpha)}{d\tau} = \frac{\lambda\sigma}{2\epsilon}(e^{-\mu\tau}\alpha - e^{-\bar{\mu}\tau}\bar{\alpha}) - \frac{\sigma}{2}e^{-\sigma\tau}|\alpha|^2$$

and the solution to this equation is

$$g_{\lambda,\tau}(\alpha) = -\frac{|\alpha|^2}{2}(1 - e^{-\sigma\tau}) + \frac{\lambda\sigma}{2\epsilon\mu}(1 - e^{-\mu\tau})\alpha - \frac{\lambda\sigma}{2\epsilon\bar{\mu}}(1 - e^{-\bar{\mu}\tau})\bar{\alpha}.$$

Putting the solutions $T_{\lambda,\tau}(\alpha)$ and $g_{\lambda,\tau}(\tau)$ into the expression for $\gamma_{\lambda,\tau}(W(\alpha))$ (4.4.5) we get the following

$$\gamma_{\lambda,\tau}(W(\alpha)) = e^{-\frac{|\alpha|^2}{2}(1-e^{-\sigma\tau})} e^{\frac{\lambda\sigma}{2\epsilon\mu}(1-e^{-\mu\tau})\alpha - \frac{\lambda\sigma}{2\epsilon\bar{\mu}}(1-e^{-\bar{\mu}\tau})\bar{\alpha}} W(e^{-\mu\tau}\alpha).$$

In order to calculate $\mathcal{L}_\sigma^*(W(\alpha))$ we use (4.3.5) equation. The first term has the following form

$$\begin{aligned} & e^{-\lambda(b^*-b)/\epsilon} \gamma_{\lambda,\tau}(e^{\lambda(b^*-b)/\epsilon} W(\alpha) e^{-\frac{\lambda}{\epsilon}(b^*-b)}) e^{\lambda(b^*-b)/\epsilon} \\ &= e^{-\lambda/\epsilon(\alpha-\bar{\alpha})} e^{-\lambda(b^*-b)/\epsilon} \gamma_{\lambda,\tau}(W(\alpha)) e^{\lambda(b^*-b)/\epsilon} \\ &= e^{-\lambda/\epsilon(\alpha-\bar{\alpha})} e^{-\frac{|\alpha|^2}{2}(1-e^{-\sigma\tau})} e^{\frac{\lambda\sigma}{2\epsilon\mu}(1-e^{-\mu\tau})\alpha - \frac{\lambda\sigma}{2\epsilon\bar{\mu}}(1-e^{-\bar{\mu}\tau})\bar{\alpha}} e^{-\lambda/\epsilon(b^*-b)} W(e^{-\mu\tau}\alpha) e^{\lambda(b^*-b)/\epsilon} \\ &= e^{-\frac{|\alpha|^2}{2}(1-e^{-\sigma\tau})} e^{-\frac{i\lambda}{\mu}(1-e^{-\mu\tau})\alpha - \frac{i\lambda}{\bar{\mu}}(1-e^{-\bar{\mu}\tau})\bar{\alpha}} W(e^{-\mu\tau}\alpha). \end{aligned}$$

Therefore from (4.3.5) we get

$$\begin{aligned} \mathcal{L}_\sigma^*(W(\alpha)) &= p e^{-\frac{|\alpha|^2}{2}(1-e^{-\sigma\tau})} e^{-\frac{i\lambda}{\mu}(1-e^{-\mu\tau})\alpha - \frac{i\lambda}{\bar{\mu}}(1-e^{-\bar{\mu}\tau})\bar{\alpha}} W(e^{-\mu\tau}\alpha) + (1-p) e^{\frac{|\alpha|^2}{2}e^{-\sigma\tau}} W(e^{-\mu\tau}\alpha) \\ &= e^{-\frac{|\alpha|^2}{2}(1-e^{-\sigma\tau})} (p e^{-\frac{i\lambda}{\mu}(1-e^{-\mu\tau})\alpha - \frac{i\lambda}{\bar{\mu}}(1-e^{-\bar{\mu}\tau})\bar{\alpha}} + 1-p) W(e^{-\mu\tau}\alpha). \end{aligned}$$

Therefore

$$(\mathcal{L}_\sigma^*)^n(W(\alpha)) = e^{-\frac{|\alpha|^2}{2}(1-e^{-n\sigma\tau})} \prod_{k=0}^{n-1} (pe^{-\frac{i\lambda}{\mu}(1-e^{-\mu\tau})e^{-k\mu\tau}\alpha - \frac{i\lambda}{\bar{\mu}}(1-e^{-\bar{\mu}\tau})e^{k\mu\tau}\bar{\alpha}} + 1 - p)W(e^{-n\mu\tau}\alpha). \quad (4.4.9)$$

In the limit n goes to infinity $W(e^{-n\mu\tau}\alpha)$ converges weakly to $\mathbb{1}$, so the dependence of the limiting state on the initial state disappears. The characteristic function gets the form

$$\omega_{\mathcal{C},\sigma}(W(\alpha)) = e^{-\frac{|\alpha|^2}{2}} \prod_{k=0}^{n-1} (pe^{-\frac{i\lambda}{\mu}(1-e^{-\mu\tau})e^{-k\mu\tau}\alpha - \frac{i\lambda}{\bar{\mu}}(1-e^{-\bar{\mu}\tau})e^{k\mu\tau}\bar{\alpha}} + 1 - p)W(e^{-n\mu\tau}\alpha). \quad (4.4.10)$$

To see the convergence of the product let us denote

$$h_k(\alpha) = p(e^{-\frac{i\lambda}{\mu}(1-e^{-\mu\tau})e^{-k\mu\tau}\alpha - \frac{i\lambda}{\bar{\mu}}(1-e^{-\bar{\mu}\tau})e^{k\mu\tau}\bar{\alpha}} - 1).$$

The product $\prod_{k=0}^{\infty}(1+h_k(\alpha))$ converges if and only if the sum $\sum_{k=0}^{\infty}|h_k(\alpha)|$ converges.

Writing $h_k(\alpha)$ as a sum of real and imaginary parts we get the following bound

$$|h_k(\alpha)| \leq 2p \frac{\lambda|\alpha|}{|\mu|^2} \left(\frac{\sigma}{2} + \epsilon \right) (1 + e^{-\frac{\sigma}{2}\tau}) e^{-k\frac{\sigma}{2}\tau}, \quad (4.4.11)$$

from which the convergence of the series immediately follows. \square

Remark 5. From (4.4.10) and (4.4.11) we see that the state $\omega_{\mathcal{C},\sigma}(\cdot)$ is regular (see Section 2.1.3), since the function $a \mapsto \omega_{\mathcal{C},\sigma}(W(a\alpha))$ is C^∞ -smooth on the vicinity of $a = 0$. By the Araki-Segal theorem (see Theorem 1), any regular state on $CCR(\mathcal{H}_{\mathcal{C}})$ is uniquely defined by the characteristic functional $\{\alpha \mapsto \omega_{\mathcal{C},\sigma}(W(\alpha))\}_{\alpha \in \mathbb{C}}$. To check the conditions of this theorem we notice that the evolution of the Weyl operator for the time $t = n\tau$ can be written as a convex combination of quasi-free completely positive

maps:

$$\begin{aligned}
(\mathcal{L}_\sigma^*)^n(W(\alpha)) &= W(e^{-n\mu\tau}\alpha) \exp\left(-\frac{|\alpha|^2}{2}(1 - e^{-n\sigma\tau})\right) \\
&\times \sum_{m=0}^{n-1} p^m (1-p)^{n-1-m} \sum_{1 \leq k_1 < \dots < k_m \leq n-1} \\
&\times \exp\left(-\frac{i\lambda}{|\mu|^2} (\bar{\mu}(1 - e^{-\mu\tau})(e^{-k_1\mu\tau} + \dots + e^{-k_m\mu\tau})\alpha - \text{c.c.})\right).
\end{aligned}$$

Here *c.c.* stands for a complex conjugate of the first term. From here we see that the limiting state $\omega_{C,\sigma}$ is an infinite combination of quasi-free states on the Weyl algebra and, in general, is not quasi-free itself.

4.5 Energy flux and entropy production

4.5.1 Energy variation in perfect cavity

Since time-dependent interaction in (4.1.2) is piece-wise constant, our system is autonomous on each interval $[(n-1)\tau, n\tau)$. Therefore, there is no variation of energy on this interval although it may jump, when a new atom enters into the cavity. Note that although the total energy corresponding to the infinite system (4.1.3) has no sense, its variation is well-defined.

Let the n -th atom is actually traveling in the cavity, i.e. $t = n(t)\tau + \nu(t)$, with $n(t) = n-1$ and $\nu(t) \in [0, \tau)$, see (4.1.11). Then one can compare the expectation of total energy of the system for the moment $t_n = (n-1)\tau + \nu(t_n)$, with that, when the $n-1$ -th atom was in the cavity, $t_{n-1} = (n-2)\tau + \nu(t_{n-1})$. Then by (4.1.3), (4.1.4) and (4.1.12) the energy variation between two moments t_{n-1} and t_n is defined

by

$$\Delta\mathcal{E}(t_n, t_{n-1}) := \text{Tr}(\rho_{\mathcal{S}}(t_n)H(t_n)) - \text{Tr}(\rho_{\mathcal{S}}(t_{n-1})H(t_{n-1})). \quad (4.5.1)$$

Lemma 6. *For any $\nu \in [0, \tau]$ one has*

$$\text{Tr}(\rho_{\mathcal{S}}((n-1)\tau + \nu)H_n) = \text{Tr}(\rho_{\mathcal{S}}((n-1)\tau)H_n). \quad (4.5.2)$$

Proof.

$$\begin{aligned} \text{Tr}(\rho_{\mathcal{S}}((n-1)\tau + \nu)H_n) &= \text{Tr}(e^{-i\nu H_n} \rho_{\mathcal{S}}((n-1)\tau) e^{i\nu H_n} H_n) \\ &= \text{Tr}(\rho_{\mathcal{S}}((n-1)\tau)H_n). \end{aligned}$$

□

Therefore

$$\Delta\mathcal{E}(t_n, t_{n-1}) = \text{Tr}(\rho_{\mathcal{S}}((n-1)\tau)[H_n - H_{n-1}]) , \quad (4.5.3)$$

Let us introduce operator

$$\pi_n(\tau) := e^{i\tau H_n} \dots e^{i\tau H_1} . \quad (4.5.4)$$

Since (4.1.4) implies

$$H_n - H_{n-1} = \lambda (b^* + b) \otimes (\eta_n - \eta_{n-1}) + \mathbb{1} \otimes E(\eta_n - \eta_{n-1}) , \quad (4.5.5)$$

by (4.5.3), (4.5.4) and by $[H_{k'}, \eta_k] = 0$ we obtain

$$\Delta \mathcal{E}(t_n, t_{n-1}) = \quad (4.5.6)$$

$$\begin{aligned} & \text{Tr}\{\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}} \pi_{n-1}(\tau)(\lambda(b^* + b) \otimes \mathbb{1})\pi_{n-1}^*(\tau)[\mathbb{1} \otimes (\eta_n - \eta_{n-1})]\} \\ & + \text{Tr}\{\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}}(\mathbb{1} \otimes E(\eta_n - \eta_{n-1}))\} . \end{aligned}$$

Lemma 7. *For any $n \geq 1$ one gets:*

$$\begin{aligned} \pi_n(\tau)(b^* \otimes \mathbb{1})\pi_n^*(\tau) &= e^{ni\tau\epsilon}(b^* \otimes \mathbb{1}) - \frac{\lambda}{\epsilon}(1 - e^{i\tau\epsilon}) \sum_{k=1}^n e^{(k-1)i\tau\epsilon} \mathbb{1} \otimes \eta_k , \\ \pi_n(\tau)(b \otimes \mathbb{1})\pi_n^*(\tau) &= e^{-ni\tau\epsilon}(b \otimes \mathbb{1}) - \frac{\lambda}{\epsilon}(1 - e^{-i\tau\epsilon}) \sum_{k=1}^n e^{-(k-1)i\tau\epsilon} \mathbb{1} \otimes \eta_k . \end{aligned} \quad (4.5.7)$$

Proof. We prove the Lemma by induction. Let

$$B_k^*(\tau) := e^{i\tau H_k}(b^* \otimes \mathbb{1})e^{-i\tau H_k} , \quad k \geq 1 . \quad (4.5.8)$$

Then by (4.1.4) the operator (4.5.8) is solution of equation

$$\partial_s B_k^*(s) = i[H_k, B_k^*(s)] = i\epsilon B_k^*(s) + \lambda \mathbb{1} \otimes \eta_k , \quad B_k^*(0) = b^* \otimes \mathbb{1} ,$$

which has the following explicit form:

$$B_k^*(\tau) = e^{i\tau\epsilon}(b^* \otimes \mathbb{1}) - \frac{\lambda}{\epsilon}(1 - e^{i\tau\epsilon}) \mathbb{1} \otimes \eta_k . \quad (4.5.9)$$

Similarly one obtains

$$B_k(\tau) = e^{-i\tau\epsilon}(b \otimes \mathbb{1}) - \frac{\lambda}{\epsilon}(1 - e^{-i\tau\epsilon}) \mathbb{1} \otimes \eta_k . \quad (4.5.10)$$

Suppose that (4.5.7) is true for π_n , we prove it for π_{n+1} .

$$\begin{aligned}
\pi_{n+1}(\tau)(b^* \otimes \mathbb{1}) &= e^{i\tau H_{n+1}}(\pi_n(\tau)(b^* \otimes \mathbb{1}))e^{-i\tau H_{n+1}} \\
&= e^{i\tau H_{n+1}}\left(e^{ni\tau\epsilon}(b^* \otimes \mathbb{1}) - \frac{\lambda}{\epsilon}(1 - e^{i\tau\epsilon}) \sum_{k=1}^n e^{(k-1)i\tau\epsilon} \mathbb{1} \otimes \eta_k\right)e^{-i\tau H_{n+1}} \\
&= e^{ni\tau\epsilon}e^{i\tau H_{n+1}}(b^* \otimes \mathbb{1})e^{-i\tau H_{n+1}} - \frac{\lambda}{\epsilon}(1 - e^{i\tau\epsilon}) \sum_{k=1}^n e^{(k-1)i\tau\epsilon}e^{i\tau H_{n+1}}(\mathbb{1} \otimes \eta_k)e^{-i\tau H_{n+1}}.
\end{aligned}$$

By (4.5.9) and by $[H_{k'}, \eta_k] = 0$ we obtain

$$\begin{aligned}
\pi_{n+1}(\tau)(b^* \otimes \mathbb{1}) &= e^{i(n+1)\tau\epsilon}(b^* \otimes \mathbb{1}) - \frac{\lambda}{\epsilon}(1 - e^{i\tau\epsilon})e^{ni\tau\epsilon} \mathbb{1} \otimes \eta_n \\
&\quad - \frac{\lambda}{\epsilon}(1 - e^{i\tau\epsilon}) \sum_{k=1}^n e^{(k-1)i\tau\epsilon} \mathbb{1} \otimes \eta_k \\
&= e^{(n+1)i\tau\epsilon}(b^* \otimes \mathbb{1}) - \frac{\lambda}{\epsilon}(1 - e^{i\tau\epsilon}) \sum_{k=1}^{n+1} e^{(k-1)i\tau\epsilon} \mathbb{1} \otimes \eta_k.
\end{aligned}$$

Since the similar formula can be obtained for $\pi_{n+1}(b \otimes \mathbb{1})$ the last formula proves the lemma.

□

Recall that we suppose that atomic beam is homogeneous, i.e., $p = \text{Tr}\{\rho_C \otimes \rho_A(\mathbb{1} \otimes \eta_n)\}$ is the probability that atom is in its excited state and p is independent of n . Then

$$\text{Tr}_{\mathcal{H}_A}\{\rho_A(\eta_{m_1}\eta_{n_2})\} = \delta_{n_1, n_2} p + (1 - \delta_{n_1, n_2}) p^2. \quad (4.5.11)$$

Then the second term in the right-hand side of (4.5.6) vanishes. Since we also supposed that the initial cavity state ρ_C is gauge-invariant, by Lemma 7 and (4.5.11)

one obtains a bound for the first term in the right-hand side of (4.5.6):

$$\begin{aligned}
& \text{Tr}\{\rho_{\mathcal{C}} \otimes \rho_{\mathcal{A}} \pi_{n-1}(\tau)(\lambda(b^* + b) \otimes \mathbb{1})\pi_{n-1}^*(\tau)[\mathbb{1} \otimes (\eta_n - \eta_{n-1})]\} \\
&= -\frac{\lambda^2}{\epsilon}(1 - e^{i\tau\epsilon}) \sum_{k=1}^{n-1} e^{(k-1)i\tau\epsilon} \text{Tr}_{\mathcal{H}_{\mathcal{A}}} \eta_k(\eta_n - \eta_{n-1}) \\
&\quad - \frac{\lambda^2}{\epsilon}(1 - e^{-i\tau\epsilon}) \sum_{k=1}^{n-1} e^{-(k-1)i\tau\epsilon} \text{Tr}_{\mathcal{H}_{\mathcal{A}}} \eta_k(\eta_n - \eta_{n-1}) \\
&= 2 \frac{\lambda^2}{\epsilon} p(1-p) [\cos((n-2)\tau\epsilon) - \cos((n-1)\tau\epsilon)] .
\end{aligned} \tag{4.5.12}$$

Hence formulae (4.5.6) and (4.5.12) prove for the total-energy variation the following statement.

Theorem 10. *The energy variation (4.5.1) between two moments t_{n-1} and t_n , where $n \geq 2$ is*

$$\Delta\mathcal{E}(t_n, t_{n-1}) = 2 \frac{\lambda^2}{\epsilon} p(1-p) [\cos((n-2)\tau\epsilon) - \cos((n-1)\tau\epsilon)] . \tag{4.5.13}$$

For the total variation between t_1 and $t_n \geq t_1$ we obtain:

$$\Delta\mathcal{E}(t_n, t_1) = \sum_{k=2}^n \Delta\mathcal{E}(t_k, t_{k-1}) = 2 \frac{\lambda^2}{\epsilon} p(1-p) [1 - \cos((n-1)\tau\epsilon)] . \tag{4.5.14}$$

4.5.2 Energy variation in leaking cavity

In the open cavity the photon energy is not piece-wise constant. Let $t_n = (n-1)\tau + \nu(t_n)$ and $t_{n-1} = (n-2)\tau + \nu(t_{n-1})$. The energy difference between the times t_{n-1} and t_n can be divided into three parts: the energy difference between the time t_{n-1} and the latest time when $(n-1)$ -st atom is still present in the cavity, the energy jump when $(n-1)$ -st atom leaves the cavity and n -th atom enters the cavity and the

energy difference between the times when the n -th atom is first present in the cavity and the time t_n :

$$\Delta E(t_n, t_{n-1}) = \Delta E((n-1)\tau, t_{n-1}) + \Delta E((n-1)\tau) + \Delta E(t_n, (n-1)\tau). \quad (4.5.15)$$

By (4.1.26) the first term can be written as follows

$$\begin{aligned} \Delta E((n-1)\tau, t_{n-1}) &= \text{Tr}(\rho_S((n-1)\tau)H_{n-1}) - \text{Tr}(\rho_S(t_{n-1})H_{n-1}) \\ &= \text{Tr}(e^{\tau L_{\sigma, n-1}}(\rho_S((n-2)\tau))H_{n-1}) - \text{Tr}(e^{\nu(t_{n-1})L_{\sigma, n-1}}(\rho_S((n-2)\tau))H_{n-1}). \end{aligned}$$

Lemma 8. *For any $0 \leq \delta \leq \tau$*

$$\begin{aligned} \text{Tr}(e^{\delta L_{\sigma, n}}(\rho_S((n-1)\tau))H_n) &= \epsilon e^{-\sigma\delta} N_{\sigma}((n-1)\tau) + p \frac{\lambda^2}{\epsilon} e^{-\sigma\delta} \\ &+ p \frac{\lambda^2 \sigma}{|\mu|^2} e^{-\frac{\sigma}{2}\delta} \sin \epsilon\delta + p \frac{\lambda^2 \sigma^2}{4\epsilon |\mu|^2} (1 - e^{-\sigma\delta}) + p(E - \frac{\lambda^2}{\epsilon^2}). \end{aligned} \quad (4.5.16)$$

Proof. Using the operator defined in (4.3.4) the last expression can be written as follows

$$\begin{aligned} &\text{Tr}(e^{\delta L_{\sigma, n}}(\rho_S((n-1)\tau))H_n) \\ &= \text{Tr}(\rho_S((n-1)\tau)e^{\delta L_{\sigma, n}^*}(H_n)) \\ &= \text{Tr}(\rho_S((n-1)\tau)e^{-iV_n}e^{\delta \tilde{L}_{\sigma, n}^*}(e^{iV_n}H_n e^{-iV_n})e^{iV_n}). \end{aligned} \quad (4.5.17)$$

From (4.2.5) we have that $e^{iV_n}H_n e^{-iV_n} = \epsilon b^*b + (E - \frac{\lambda^2}{\epsilon^2})\eta_n$.

The action of the operator (4.3.4) can be written explicitly using (4.2.4). Then

continuing the calculation of (4.5.17) we obtain

$$\begin{aligned}
& \text{Tr}(\rho_S((n-1)\tau)e^{-\lambda(b^*-b)/\epsilon}e^{\delta(L_\lambda^C)^*}(\epsilon b^*b)e^{\lambda(b^*-b)/\epsilon} \otimes \eta_n) \\
& + \text{Tr}(\rho_S((n-1)\tau)e^{\delta(L_0^C)^*}(\epsilon b^*b) \otimes (1-\eta_n)) \\
& + \text{Tr}(\rho_S((n-1)\tau)\mathbb{1} \otimes (E - \frac{\lambda^2}{\epsilon^2})\eta_n) \\
& = p\text{Tr}_C(\rho_C((n-1)\tau)e^{-\lambda(b^*-b)/\epsilon}e^{\delta(L_\lambda^C)^*}(\epsilon b^*b)e^{\lambda(b^*-b)/\epsilon}) \\
& + (1-p)\epsilon e^{-\sigma\delta}\text{Tr}_C(\rho_C((n-1)\tau)b^*b) + p(E - \frac{\lambda^2}{\epsilon^2}),
\end{aligned} \tag{4.5.18}$$

by (4.3.13) for $\lambda = 0$ we obtain $e^{\delta(L_0^C)^*}(b^*b) = e^{-\sigma\delta}b^*b$. To calculate the first term in (4.5.18) with $\lambda \neq 0$ we note that

$$(L_\lambda^C)^*(b^*b) = -\sigma b^*b + \frac{\lambda\sigma}{2\epsilon}(b^* + b).$$

Then we get a differential equation for $\gamma_{\lambda,\tau} = e^{\delta(L_\lambda^C)^*}$:

$$\frac{d\gamma_{\lambda,\delta}(b^*b)}{d\delta} = -\sigma\gamma_{\lambda,\delta}(b^*b) + \frac{\lambda\sigma}{2\epsilon}(\gamma_{\lambda,\delta}(b) + \gamma_{\lambda,\delta}(b^*)). \tag{4.5.19}$$

From (4.3.11) and (4.3.12) we find

$$\gamma_{\lambda,\delta}(b) + \gamma_{\lambda,\delta}(b^*) = e^{-\mu\delta}b + \epsilon^{-\bar{\mu}\delta}b^* + \frac{\lambda\sigma}{2\epsilon\bar{\mu}}(1 - e^{-\mu\delta}) + \frac{\lambda\sigma}{2\epsilon\bar{\mu}}(1 - e^{-\bar{\mu}\delta}).$$

The the solution of (4.5.19) yields

$$\begin{aligned}
\gamma_{\lambda,\delta}(b^*b) &= e^{-\sigma\delta}b^*b - \frac{\lambda\sigma}{2\epsilon\bar{\mu}}(e^{-\sigma\delta} - e^{-\mu\delta})b - \frac{\lambda\sigma}{2\epsilon\bar{\mu}}(e^{-\sigma\delta} - e^{-\bar{\mu}\delta})b^* \\
&+ \frac{\lambda^2\sigma^2}{4\epsilon^2|\mu|^2}(1 + e^{-\sigma\delta} - 2e^{-\frac{\sigma}{2}\delta}\cos\epsilon\delta).
\end{aligned}$$

After the shift transformation, $b \rightarrow b + \frac{\lambda}{\epsilon}$, in the last formula we obtain for the gauge-invariant initial state $\rho_{\mathcal{C}}$ the $\lambda \neq 0$ term

$$\begin{aligned} & \text{Tr}_{\mathcal{C}}(\rho_{\mathcal{C}}((n-1)\tau)e^{-\lambda(b^*-b)/\epsilon}e^{\delta(L_{\lambda}^{\mathcal{C}})^*}(\epsilon b^*b)e^{\lambda(b^*-b)/\epsilon}) \\ &= \epsilon e^{-\sigma\delta} \text{Tr}_{\mathcal{C}}(\rho_{\mathcal{C}}((n-1)\tau)b^*b) + e^{-\sigma\delta} \frac{\lambda^2}{\epsilon} - \frac{\lambda^2\sigma}{2\epsilon\bar{\mu}}(e^{-\sigma\delta} - e^{-\mu\delta}) - \frac{\lambda^2\sigma}{2\epsilon\mu}(e^{-\sigma\delta} - e^{-\bar{\mu}\delta}) \\ &+ \frac{\lambda^2\sigma^2}{4\epsilon|\mu|^2}(1 + e^{-\sigma\delta} - 2e^{-\frac{\sigma}{2}\delta} \cos \epsilon\delta). \end{aligned} \quad (4.5.20)$$

Since $\text{Tr}_{\mathcal{C}}(\rho_{\mathcal{C}}((n-1)\tau)b^*b) = N_{\sigma}((n-1)\tau)$ is the number of photons in the cavity at the moment $t = (n-1)\tau$, by plugging the above expression (4.5.20) into (4.5.18) we find (4.5.16). \square

With the help of this lemma we find the energy variance

$$\begin{aligned} \Delta E((n-1)\tau, t_{n-1}) &= \text{Tr}(\rho_S((n-1)\tau)H_{n-1}) - \text{Tr}(\rho_S(t_{n-1})H_{n-1}) \\ &= \text{Tr}(e^{\tau L_{\sigma, n-1}}(\rho_S((n-2)\tau))H_{n-1}) - \text{Tr}(e^{\nu(t_{n-1})L_{\sigma, n-1}}(\rho_S((n-2)\tau))H_{n-1}) \\ &= \epsilon(e^{-\sigma\tau} - e^{-\sigma\nu(t_{n-1})})N_{\sigma}((n-2)\tau) + p\frac{\lambda^2}{\epsilon}(e^{-\sigma\tau} - e^{-\sigma\nu(t_{n-1})}) \\ &- p\frac{\lambda^2}{2\epsilon|\mu|^2}\frac{\sigma^2}{2}(e^{-\sigma\tau} - e^{-\sigma\nu(t_{n-1})}) + p\frac{\lambda^2\sigma}{|\mu|^2}(e^{-\frac{\sigma}{2}\tau} \sin \epsilon\tau - e^{-\frac{\sigma}{2}\nu(t_{n-1})} \sin \epsilon\nu(t_{n-1})), \end{aligned} \quad (4.5.21)$$

which is the first term in the energy difference (4.5.15).

The third term in (4.5.15) can be calculated similarly

$$\begin{aligned}
\Delta E(t_n, (n-1)\tau) &= \text{Tr}(\rho_S(t_n)H_n) - \text{Tr}(\rho_S((n-1)\tau)H_n) \\
&= \text{Tr}(e^{\nu(t_n)L_{\sigma,n}}(\rho_S((n-1)\tau))H_n) - \text{Tr}(\rho_S((n-1)\tau)H_n) \\
&= \epsilon(e^{-\sigma\nu(t_n)} - 1)N_\sigma((n-1)\tau) - p\frac{\lambda^2}{\epsilon}(1 - e^{-\sigma\nu(t_n)}) + p\frac{\lambda^2\sigma}{|\mu|^2}e^{-\frac{\sigma}{2}\nu(t_n)}\sin \epsilon\nu(t_n) \\
&\quad + p\frac{\lambda^2}{2\epsilon|\mu|^2}\frac{\sigma^2}{2}(1 - e^{-\sigma\nu(t_n)}).
\end{aligned} \tag{4.5.22}$$

To calculate the second term in (4.5.15) note that

$$\begin{aligned}
\Delta E((n-1)\tau) &= \text{Tr}(\rho_S((n-1)\tau)H_{n-1}) - \text{Tr}(\rho_S((n-1)\tau)H_n) \\
&= \text{Tr}(e^{\tau L_{\sigma,n-1}} \dots e^{\tau L_{\sigma,1}}(\rho_C \otimes \rho_A)(H_n - H_{n-1})) \\
&= \text{Tr}(U_{n-1}^\sigma(\rho_C \otimes \rho_A)(H_n - H_{n-1})) \\
&= \text{Tr}(\rho_C \otimes \rho_A(U_{n-1}^\sigma)^*(\lambda(b^* + b) \otimes (\eta_n - \eta_{n-1}) + \mathbb{1} \otimes E(\eta_n - \eta_{n-1}))) \\
&= \text{Tr}(\rho_C \otimes \rho_A(U_{n-1}^\sigma)^*(\lambda(b^* + b) \otimes (\eta_n - \eta_{n-1}))) \\
&= \text{Tr}\{\rho_C \otimes \rho_A(U_{n-1}^\sigma)^*(\lambda(b^* + b) \otimes \mathbb{1})[\mathbb{1} \otimes (\eta_n - \eta_{n-1})]\},
\end{aligned}$$

where $U_n^\sigma = U_{t=n\tau, s=0}^\sigma = e^{\tau L_{\sigma,n}} \dots e^{\tau L_{\sigma,1}}$ was defined in (4.1.24).

Lemma 9.

$$(U_n^\sigma)^*(b \otimes \mathbb{1}) = e^{-n\mu\tau}b \otimes \mathbb{1} - \frac{\lambda i}{\mu}(1 - e^{-\mu\tau}) \sum_{k=1}^n e^{-(k-1)\mu\tau} \mathbb{1} \otimes \eta_k.$$

Proof. We prove this lemma by induction. Suppose the formula is true for $(U_n^\sigma)^*$ we show it is true for $(U_{n+1}^\sigma)^*$. The same line of reasoning as in (4.3.4) can be used to

write the action of the operator $e^{\tau L_{\sigma,n}^*}$

$$\begin{aligned} (U_{n+1}^\sigma)^*(b \otimes \mathbb{1}) &= e^{\tau L_{\sigma,n+1}^*}((U_n^\sigma)^*(b \otimes \mathbb{1})) \\ &= e^{-iV_{n+1}} e^{\tau \tilde{L}_{\sigma,n+1}^*} (e^{iV_{n+1}} (e^{-n\mu\tau} b \otimes \mathbb{1} - \frac{\lambda i}{\mu} (1 - e^{-\mu\tau}) \sum_{k=1}^n e^{-(k-1)\mu\tau} \mathbb{1} \otimes \eta_k) e^{-iV_{n+1}}) e^{iV_{n+1}}. \end{aligned}$$

By the shift transformation we find the right-hand side

$$\begin{aligned} &= e^{-iV_{n+1}} e^{\tau \tilde{L}_{\sigma,n+1}^*} (e^{-n\mu\tau} (b - \frac{\lambda}{\epsilon}) \otimes \eta_{n+1} + e^{-n\mu\tau} b \otimes (1 - \eta_{n+1})) e^{iV_{n+1}} \\ &\quad - \frac{\lambda i}{\mu} (1 - e^{-\mu\tau}) \sum_{k=1}^n e^{-(k-1)\mu\tau} \otimes \eta_k. \end{aligned}$$

Applying (4.3.11) to the first line above we find that

$$\begin{aligned} e^{-iV_{n+1}} e^{\tau \tilde{L}_{\sigma,n+1}^*} ((b - \frac{\lambda}{\epsilon}) \otimes \eta_{n+1}) e^{iV_{n+1}} &= (e^{-\mu\tau} b - \frac{\lambda i}{\mu} (1 - e^{-\mu\tau})) \otimes \eta_{n+1}, \\ e^{-iV_{n+1}} e^{\tau \tilde{L}_{\sigma,n+1}^*} (b \otimes (1 - \eta_{n+1})) e^{iV_{n+1}} &= e^{-\mu\tau} b \otimes (1 - \eta_{n+1}). \end{aligned}$$

Consequently,

$$\begin{aligned} (U_{n+1}^\sigma)^*(b \otimes \mathbb{1}) &= e^{-n\mu\tau} (e^{-\mu\tau} b - \frac{\lambda i}{\mu} (1 - e^{-\mu\tau})) \otimes \eta_{n+1} \\ &\quad + e^{-n\mu\tau} e^{-\mu\tau} b \otimes (1 - \eta_{n+1}) - \frac{\lambda i}{\mu} (1 - e^{-\mu\tau}) \sum_{k=1}^n e^{-(k-1)\mu\tau} \otimes \eta_k \\ &= e^{-(n+1)\mu\tau} b \otimes \mathbb{1} - \frac{\lambda i}{\mu} (1 - e^{-\mu\tau}) \sum_{k=1}^{n+1} e^{-(k-1)\mu\tau} \otimes \eta_k, \end{aligned}$$

which proves the lemma. \square

Therefore, the energy jump at the moment $t = (n - 1)\tau$ is

$$\begin{aligned}\Delta E((n - 1)\tau) &= \text{Tr}\{\rho_C \otimes \rho_A(U_{n-1}^\sigma)^*(\lambda(b^* + b) \otimes \mathbb{1})[\mathbb{1} \otimes (\eta_n - \eta_{n-1})]\} \\ &= -\frac{\lambda^2 i}{\mu}(1 - e^{-\mu\tau}) \sum_{k=1}^{n-1} e^{-(k-1)\mu\tau} \text{Tr}(\rho_A \eta_k(\eta_n - \eta_{n-1})) \\ &\quad + \frac{\lambda^2 i}{\bar{\mu}}(1 - e^{-\bar{\mu}\tau}) \sum_{k=1}^{n-1} e^{-(k-1)\bar{\mu}\tau} \text{Tr}(\rho_A \eta_k(\eta_n - \eta_{n-1})).\end{aligned}$$

The Bernoulli property (4.5.11) implies

$$\begin{aligned}\Delta E((n - 1)\tau) &= \frac{\lambda^2 i}{\mu}(1 - e^{-\mu\tau})p(1 - p)e^{-(n-2)\mu\tau} - \frac{\lambda^2 i}{\bar{\mu}}(1 - e^{-\bar{\mu}\tau})p(1 - p)e^{-(n-2)\bar{\mu}\tau} \\ &= 2\lambda^2 p(1 - p) \frac{4\epsilon}{\sigma^2 + 4\epsilon^2} (e^{-(n-2)\frac{\sigma}{2}\tau} \cos(n - 2)\epsilon\tau - e^{-(n-1)\frac{\sigma}{2}\tau} \cos(n - 1)\epsilon\tau) \quad (4.5.23) \\ &\quad + 4\lambda^2 p(1 - p) \frac{\sigma}{\sigma^2 + 4\epsilon^2} (e^{-(n-2)\frac{\sigma}{2}\tau} \sin(n - 2)\epsilon\tau - e^{-(n-1)\frac{\sigma}{2}\tau} \sin(n - 1)\epsilon\tau).\end{aligned}$$

The energy difference between points t_n and t_{n-1} is the sum all the three terms (4.5.21-4.5.23).

Theorem 11. *The energy variation between two moments $t_{n-1} = (n - 2)\tau + \nu(t_{n-1})$ and $t_n = (n - 1)\tau + \nu(t_n)$, where $n \geq 2$ is equal to*

$$\begin{aligned}\Delta E(t_n, t_{n-1}) &= \epsilon(e^{-\sigma\tau} - e^{-\sigma\nu(t_{n-1})})N_\sigma((n - 2)\tau) + \epsilon(e^{-\sigma\nu(t_n)} - 1)N_\sigma((n - 1)\tau) \\ &\quad - p \frac{\lambda^2 \epsilon}{|\mu|^2} (1 - e^{-\sigma\tau} + e^{-\sigma\nu(t_{n-1})} - e^{-\sigma\nu(t_n)}) \quad (4.5.24)\end{aligned}$$

$$+ p \frac{\lambda^2 \sigma}{|\mu|^2} (e^{-\frac{\sigma}{2}\tau} \sin \epsilon\tau + e^{-\frac{\sigma}{2}\nu(t_n)} \sin \epsilon\nu(t_n) - e^{-\frac{\sigma}{2}\nu(t_{n-1})} \sin \epsilon\nu(t_{n-1})) \quad (4.5.25)$$

$$+ 2\lambda^2 p(1 - p) \frac{4\epsilon}{\sigma^2 + 4\epsilon^2} (e^{-(n-2)\frac{\sigma}{2}\tau} \cos(n - 2)\epsilon\tau - e^{-(n-1)\frac{\sigma}{2}\tau} \cos(n - 1)\epsilon\tau) \quad (4.5.26)$$

$$+ 4\lambda^2 p(1 - p) \frac{\sigma}{\sigma^2 + 4\epsilon^2} (e^{-(n-2)\frac{\sigma}{2}\tau} \sin(n - 2)\epsilon\tau - e^{-(n-1)\frac{\sigma}{2}\tau} \sin(n - 1)\epsilon\tau). \quad (4.5.27)$$

By (4.5.24) we obtain for the total variation between $t_1 = 0$ and $t_n > t_1$:

$$\begin{aligned}
\Delta E(t_n, t_1) &= \sum_{k=2}^n \Delta E(t_k, t_{k-1}) \\
&= \epsilon(e^{-\sigma\nu(t_n)} - 1)N_\sigma((n-1)\tau) + \epsilon(e^{-\sigma\tau} - 1) \sum_{k=2}^n N_\sigma((k-2)\tau) \\
&\quad - (n-1)p \frac{\lambda^2 \epsilon}{|\mu|^2} (1 - e^{-\sigma\tau}) - p \frac{\lambda^2 \epsilon}{|\mu|^2} (1 - e^{-\sigma\nu(t_n)}) \\
&\quad + (n-1)p \frac{\lambda^2 \sigma}{|\mu|^2} e^{-\frac{\sigma}{2}\tau} \sin \epsilon\tau + p \frac{\lambda^2 \sigma}{|\mu|^2} e^{-\frac{\sigma}{2}\nu(t_n)} \sin \epsilon\nu(t_n) \\
&\quad + 2\lambda^2 p(1-p) \frac{4\epsilon}{\sigma^2 + 4\epsilon^2} (e^{-(n-2)\frac{\sigma}{2}\tau} \cos(n-2)\epsilon\tau - e^{-(n-1)\frac{\sigma}{2}\tau} \cos(n-1)\epsilon\tau) \\
&\quad + 4\lambda^2 p(1-p) \frac{\sigma}{\sigma^2 + 4\epsilon^2} (e^{-(n-2)\frac{\sigma}{2}\tau} \sin(n-2)\epsilon\tau - e^{-(n-1)\frac{\sigma}{2}\tau} \sin(n-1)\epsilon\tau) .
\end{aligned}$$

4.5.3 Entropy production in perfect cavity

In this section, we make contact with thermodynamics of systems out of equilibrium, and in particular with the second law of thermodynamics, see [3]-[5].

Let ρ and ρ_0 be two normal states on algebra $\mathfrak{A}(\mathcal{H})$. We define the *relative entropy* $\text{Ent}(\rho|\rho_0)$ of the state ρ with respect to ρ_0 by

$$\text{Ent}(\rho|\rho_0) := \text{Tr}_{\mathcal{H}}(\rho \ln \rho - \rho \ln \rho_0) \geq 0 , \quad (4.5.28)$$

where non-negativity follows from the *Jensen inequality*: $\text{Tr}_{\mathcal{H}}(\rho \ln \mathcal{A}) \leq \ln \text{Tr}_{\mathcal{H}}(\rho \mathcal{A})$, applied to observable $\mathcal{A} = \rho_0/\rho$.

In the case of the non-leaking cavity we have:

$$\Delta \mathcal{S}(t) := \text{Ent}(\rho_S(t)|\rho_S(t=0)) = \text{Tr}\{\rho_S(t)(\ln \rho_S(t) - \ln \rho_C \otimes \rho_A)\} , \quad (4.5.29)$$

where dynamics is defined by (4.1.12). Suppose that all atoms of the beam are in the

Gibbs state with temperature $1/\beta$, which formally can be written as

$$\rho_{\mathcal{A}}(\beta) := \bigotimes_{n \geq 1} \rho_{\mathcal{A}_n}(\beta) , \quad \rho_{\mathcal{A}_n}(\beta) := \frac{e^{-\beta H_{\mathcal{A}_n}}}{Z(\beta)} , \quad (4.5.30)$$

see (4.1.3). Since $\rho_S(t=0) = \rho_C \otimes \rho_{\mathcal{A}} = (\rho_C \otimes \mathbb{1})(\mathbb{1} \otimes \rho_{\mathcal{A}})$ and $\text{Tr}_{\mathcal{H}_C \otimes \mathcal{H}_{\mathcal{A}}} \{\rho_S(t) \ln \rho_S(t)\} = \text{Tr}_{\mathcal{H}_C \otimes \mathcal{H}_{\mathcal{A}}} \{\rho_S(0) \ln \rho_S(0)\}$, the relative entropy (4.5.29) gets the form

$$\begin{aligned} \Delta S(t) : &= \text{Tr}_{\mathcal{H}_C} \{[\rho_C - \rho_C^{(n)}] \ln \rho_C\} \\ &- \beta \sum_{k=1}^n \text{Tr} \{[\rho_S(0) - \rho_S(n\tau)](\mathbb{1} \otimes H_{\mathcal{A}_k})\} , \end{aligned} \quad (4.5.31)$$

for $t = n\tau + \nu$, see (4.1.11). Here $\rho_C^{(n)}$ is defined by (4.1.15).

Remark 6. For any Hamiltonian H_n that acts non-trivially on $\mathcal{H}_C \otimes \mathcal{H}_{\mathcal{A}_n}$ and any Hamiltonian $H_{\mathcal{A}_k}$ acting on $\mathcal{H}_{\mathcal{A}_k}$ we have $[H_n, H_{\mathcal{A}_k}] = 0$ for $n \neq k$. Note that by (4.1.14) for our model $[H_n, H_{\mathcal{A}_k}] = 0$ for any n, k .

Then by virtue of (4.1.15) and (4.1.16) we have

$$\begin{aligned} &\text{Tr} \{ \rho_S(n\tau) (\mathbb{1} \otimes H_{\mathcal{A}_k}) \} \\ &= \text{Tr} \{ e^{-i\tau H_n} \dots e^{-i\tau H_{k+1}} \rho_S(k\tau) e^{i\tau H_{k+1}} \dots e^{i\tau H_n} (\mathbb{1} \otimes H_{\mathcal{A}_k}) \} \\ &= \text{Tr} \{ \rho_S(k\tau) (\mathbb{1} \otimes H_{\mathcal{A}_k}) \} . \end{aligned} \quad (4.5.32)$$

By the same argument one obtains also

$$\begin{aligned} &\text{Tr} \{ \rho_S(0) (\mathbb{1} \otimes H_{\mathcal{A}_k}) \} \\ &= \text{Tr} \{ e^{-i\tau H_{k-1}} \dots e^{-i\tau H_1} \rho_S(0) e^{i\tau H_1} \dots e^{i\tau H_{k-1}} (\mathbb{1} \otimes H_{\mathcal{A}_k}) \} \\ &= \text{Tr} \{ \rho_S((k-1)\tau) (\mathbb{1} \otimes H_{\mathcal{A}_k}) \} . \end{aligned} \quad (4.5.33)$$

In the case of our model (see Remark 6) $H_{\mathcal{A}_k} = e^{i\tau H_k}(H_{\mathcal{A}_k})e^{-i\tau H_k}$. Therefore the last formula gets the form

$$\mathrm{Tr}\{\rho_{\mathcal{S}}(0) (\mathbb{1} \otimes H_{\mathcal{A}_k})\} = \mathrm{Tr}\{\rho_{\mathcal{S}}(k\tau) (\mathbb{1} \otimes H_{\mathcal{A}_k})\}. \quad (4.5.34)$$

Equations (4.5.32) and (4.5.34) shows that the second term in the entropy production (4.5.31) vanishes.

If we suppose that the initial cavity state is Gibbs state for the temperature $1/\beta$ (4.2.2), then by (4.1.20) and (4.2.3) one gets

$$\begin{aligned} \Delta S(t) &= \mathrm{Tr}_{\mathcal{H}_c}\{[\rho_c - \rho_c^{(n)}] \ln \rho_c\} \\ &= \mathrm{Tr}_{\mathcal{H}_c}\{[\rho_c - \rho_c^{(n)}](-\beta \epsilon b^* b)\} \\ &= \beta \epsilon (N(t) - N(0)) , \end{aligned} \quad (4.5.35)$$

where the photon number $N(t)$ is defined in (4.2.3).

If we denote by $\Delta \mathcal{E}^C(t) = \epsilon(N(t) - N(0))$ the energy variation of the thermal cavity defined by the photon number variation, then (4.5.35) expresses the 2nd Law of Thermodynamics

$$\Delta S(t) = \beta \Delta \mathcal{E}^C(t) , \quad (4.5.36)$$

for the pumping of cavity.

Remark 7. *In the general situation when $[H_n, H_{\mathcal{A}_n}] \neq 0$ combining (4.5.31) with (4.5.32) and (4.5.33) we get for the entropy production at the moment $t = n\tau + \nu$*

$$\begin{aligned} \Delta S(t) : &= \mathrm{Tr}_{\mathcal{H}_c}\{[\rho_c - \rho_c^{(n)}] \ln \rho_c\} \\ &+ \beta \sum_{k=1}^n \mathrm{Tr}\{[\rho_{\mathcal{S}}(k\tau) - \rho_{\mathcal{S}}((k-1)\tau)] (\mathbb{1} \otimes H_{\mathcal{A}_k})\} . \end{aligned} \quad (4.5.37)$$

The last term in (4.5.37) can be rewritten into standard form [3]-[5], if one uses the identities:

$$\begin{aligned}
Tr\{\rho_S(k\tau) (\mathbb{1} \otimes H_{\mathcal{A}_k})\} &= Tr\{e^{\tau L_k} \dots e^{\tau L_1} (\rho_C \otimes \rho_A) (\mathbb{1} \otimes H_{\mathcal{A}_k})\} \\
&= Tr\{e^{-i\tau H_k} (e^{\tau L_{k-1}} \dots e^{\tau L_1} [\rho_C \otimes \bigotimes_{n=1}^{k-1} \rho_n]) \otimes \rho_k e^{i\tau H_k} (\mathbb{1} \otimes H_{\mathcal{A}_k})\} [\mathbb{1} \otimes \bigotimes_{m>k} \rho_m] \\
&= Tr_{\mathcal{H}_C \otimes \mathcal{H}_{\mathcal{A}_k}} \{(\rho_C^{(k-1)} \otimes \rho_k) e^{i\tau H_k} (\mathbb{1} \otimes H_{\mathcal{A}_k}) e^{-i\tau H_k}\} ,
\end{aligned}$$

and

$$Tr\{\rho_S((k-1)\tau) (\mathbb{1} \otimes H_{\mathcal{A}_k})\} = Tr_{\mathcal{H}_C \otimes \mathcal{H}_{\mathcal{A}_k}} \{\rho_C^{(k-1)} \otimes \rho_k H_{\mathcal{A}_k}\} .$$

For $t = n\tau + \nu$ they yield the formula for the non-leaking entropy production

$$\begin{aligned}
\Delta S(t) &= Tr_{\mathcal{H}_C} \{[\rho_C - \rho_C^{(n)}] \ln \rho_C\} \\
&+ \beta \sum_{k=1}^n Tr_{\mathcal{H}_C \otimes \mathcal{H}_{\mathcal{A}_k}} \{(\rho_C^{(k-1)} \otimes \rho_k) [e^{i\tau H_k} (\mathbb{1} \otimes H_{\mathcal{A}_k}) e^{-i\tau H_k} - \mathbb{1} \otimes H_{\mathcal{A}_k}]\} .
\end{aligned} \tag{4.5.38}$$

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